Model Theory of Tame Classes of Finite Structures

Part 1: Locality

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Finite Model Theory

In the 1980s, the term *finite model theory* came to be used to describe the study of the *expressive power* of logics on the class of all finite structures.

The logics of interest include *first-order logic*, *second-order logic*, many in between and many beyond.

The motivation for the study comes from computer science (especially *complexity theory* and *database theory*). Many problems in these areas are naturally formulated as questions about the expressive power of logics. And, the structures involved in computation are *finite*.

The methods deployed bear only a distant relationship with *classical model theory*.

But, there is a recent *convergence* and we will review this in this tutorial.

Finite Model Theory – Early Trends

Kolaitis in a tutorial on finite model theory (**LICS 93**) identified trends in the results in the subject:

- *Negative*: showing the failure of classical model-theoretic results on finite structures.
  Compactness. Completeness. Interpolation and preservation theorems.
- *Conservative*: showing that certain classical model-theoretic results continue to hold on finite structures.
  Some consequences of compactness. Monotone vs. positive inductions. Locality.
- *Positive*: exploring concepts and results which are specific to finite structures.
  Descriptive complexity. 0–1 laws.

Compactness and Preservation

The *compactness theorem* is not available when we restrict ourselves to *finite structures*.

$$\exists x_1 \cdots \exists x_n \bigwedge_{i \neq j} x_i \neq x_j$$

The collection of these sentences has no finite model, even though every finite subset has.

Many of the consequences of compactness also fail on finite structures.

Existential Preservation
A sentence φ is equivalent to an existential sentence if, and only if, the models of φ are closed under extensions.

(Łoś-Tarski)

Proving Preservation

It is trivial to see that the syntactic restriction implies the semantic restriction.

The other direction, of *expressive completeness*, is usually proved using compactness.

For example, if φ is closed under extensions:
Take Φ to be the existential consequences of φ and show $\Phi \models \varphi$ by:

$$\begin{array}{ccc} \mathbb{A} \models \Phi \cup \{\varphi\} & \preceq & \mathbb{A}^* \\ & & \cap \\ \mathbb{B} \models \Phi \cup \{\neg\varphi\} & \preceq & \mathbb{B}^* \end{array}$$

Relativized Preservation

We consider *relativizations* of expressive completeness to classes of structure $\mathcal{C}$:

> *If φ satisfies the semantic condition restricted to $\mathcal{C}$, it is equivalent (on $\mathcal{C}$) to a sentence in the restricted syntactic form.*

Restricting the class $\mathcal{C}$ in this statement weakens both the hypothesis and the conclusion.

Łoś-Tarski is known to fail when $\mathcal{C}$ is the class of all finite structures.

(Tait)

Restricting further may make it true.

Minimal Models

For a sentence φ whose models are closed under extensions, a *minimal model* $\mathbb{A}$ is a structure such that:

- $\mathbb{A} \models \varphi$; and
- for any proper substructure $\mathbb{B} \subseteq \mathbb{A}$, $\mathbb{B} \not\models \varphi$.

φ is equivalent to an *existential sentence* on a class $\mathcal{C}$ if, and only if, φ has *finitely many* minimal models in $\mathcal{C}$.

Tame Classes

The class of *all finite structures* is not *well-behaved* in a model-theoretic sense.

Sometimes good *model-theoretic* behaviour can be recovered by restricting outselves further.

Let $\mathcal{BD}_d$ denote the class of finite graphs of *degree* at most d.

Theorem (Atserias, D., Grohe 2008)
The extension preservation theorem holds on $\mathcal{BD}_d$ for any d.

The proof uses *locality* rather than *compactness*.

Gaifman Graph

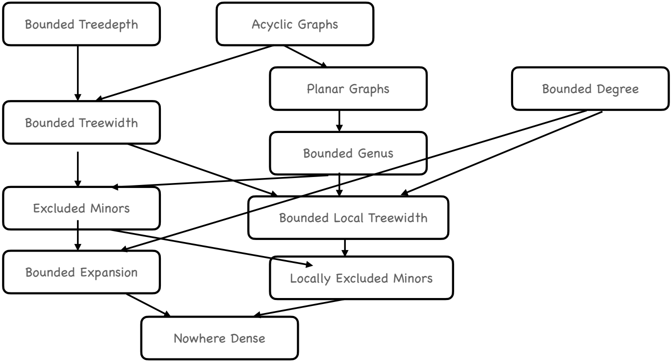
For a structure $\mathbb{A}$ in a *relational vocabulary* $\tau = (R_1, \ldots, R_r)$, the *Gaifman graph* of $\mathbb{A}$ is the undirected graph whose vertices are the elements of $\mathbb{A}$ and

> *there is an edge* $u \sim v$ *if, and only if, there is some relation* R_i *with a tuple including both* u *and* v.

The tameness conditions on a class $\mathcal{C}$ are often expressed as restrictions on the class of *Gaifman graphs* of structures in $\mathcal{C}$.

For simplicity, we usually restrict ourselves to vocabularies with one *binary* relation and a collection of *unary* relations.

Sparse Tame Classes



Algorithmic Tameness

These restricted classes of graphs have been much studied in the field of *paramaterized complexity* because of their *algorithmic tameness*.

They admit *efficient algorithms* for problems that are hard in general.

Scattered Set:
Given: a graph G and positive integers l and r
Decide: does G contain l distinct vertices that are pairwise distance at least r apart.

The problem is NP-complete. And, under reasonable assumptions, cannot be solved in time $n^{o(l)}$.

On the *tame* classes, it can be solved in time

$$f(l, r)n^c$$

First-Order Satisfaction

The *algorithmic tameness* of these classes has an explanation in *logic*

We consider the problem of evaluating first-order sentences in finite structures.

First-order evaluation:
Given: a structure $\mathbb{A}$ and a first-order sentence φ
Decide: whether $\mathbb{A} \models \varphi$.

The problem can be solved in time $O(ln^m)$, where l is the *length* of φ and n the *number of elements* of $\mathbb{A}$.
m is the nesting depth of quantifiers in φ (or by a more careful accounting, the number of distinct variables occurring in φ)

Parameterized Complexity

FPT—the class of problems of input size n and *parameter* l which can be solved in time $O(f(l)n^c)$ for some computable function f and constant c.

There is a hierarchy of *intractable* classes.

$$\mathsf{FPT} \subseteq W[1] \subseteq W[2] \subseteq \cdots \subseteq \mathsf{AW}[\star]$$

The satisfaction relation for first-order logic ($\mathbb{A} \models \varphi$), parameterized by the length of φ is $\mathsf{AW}[\star]$-complete.

Nowhere-Dense Classes

The most general result on the tractability of first-order satisfaction in *sparse* classes is:

Theorem (Grohe, Kreutzer, Siebertz)
For any nowhere dense class of graphs $\mathcal{C}$ and any $\epsilon > 0$, there is an algorithm deciding $G \models \varphi$ in time $O(f(l)n^{1+\epsilon})$ for some function f.

Here, l is the length of φ and n the number of elements in G.

Substructures

For structures $\mathbb{A} = (A, R_1^{\mathbb{A}}, \ldots, R_r^{\mathbb{A}})$ and $\mathbb{B} = (B, R_1^{\mathbb{B}}, \ldots, R_r^{\mathbb{B}})$ in a *vocabulary* τ, we say

$\mathbb{B}$ is a *substructure* of $\mathbb{A}$ if:

$B \subseteq A$; and

$R_i^{\mathbb{B}} \subseteq R_i^{\mathbb{A}}$ for all i.

$\mathbb{B}$ is an *induced substructure* of $\mathbb{A}$ if:

$B \subseteq A$; and

$R_i^{\mathbb{B}} = R_i^{\mathbb{A}} \cap A^{\mathsf{arity}(R_i)}$ for all i.

Monotone and Hereditary Classes

We say that a class $\mathcal{C}$ is

monotone if whenever $\mathbb{A}$ is in $\mathcal{C}$, so is every *substructure* of $\mathbb{A}$.

hereditary if whenever $\mathbb{A}$ is in $\mathcal{C}$, so is every *induced substructure* of $\mathbb{A}$.

Theorem (Grohe, Kreutzer, Siebertz)
For any monotone class $\mathcal{C}$, if $\mathcal{C}$ is not nowhere dense, then first-order satisfaction on $\mathcal{C}$ is $AW[\star]$-hard.

Local Formulas

We write $\delta(x, y) > d$ for the formula of FO that says that the distance between x and y is greater than d.

We write $\psi^r(x)$ to denote the formula obtained from $\psi(x)$ by relativising all quantifiers to the set $N_r = \{y \mid \delta(x, y) < r\}$, i.e.

Each subformula $\exists y\theta$ *is replaced by* $\exists y(\delta(x, y) < r) \wedge \theta^r$
Each subformula $\forall y\theta$ *is replaced by* $\forall y(\delta(x, y) < r) \rightarrow \theta^r$

Gaifman's Theorem

A *basic local sentence* is a sentence of the form

$$\exists x_1 \cdots \exists x_s \left(\bigwedge_{i \neq j} \delta(x_i, x_j) > 2r \wedge \bigwedge_i \psi^r(x_i) \right)$$

Theorem (Gaifman)
Every first-order sentence is equivalent to a Boolean combination of basic local sentences.

We call this the *Gaifman normal form* of the sentence.

Evaluating First-Order Logic

We now want to use Gaifman's theorem to establish:

Theorem (Seese)
For every sentence φ of FO and every d there is a linear time algorithm which, given a graph $G \in \mathcal{BD}_d$ determines whether $G \models \varphi$.

Indeed, $G \models \varphi$ can be decided by an algorithm running in time $f(l)n$.

First, note that the *Gaifman normal form* of φ can be computed from φ.
Thus, it suffices to prove the above for *basic local sentences*.

Neighbourhoods

Given a structure $\mathbb{A}$, an element $a \in \mathbb{A}$ and a positive integer r, we write $N_r(a)$ to denote the elements that are at distance at most r from a in $\mathbb{A}$.

For a set of elements S, $N_r(S)$ denotes $\bigcup_{a \in S} N_r(a)$.

We also write $\mathrm{Nbd}_r^{\mathbb{A}}(a)$ to denote the *r-neighbourhood* of a in $\mathbb{A}$.

That is, the substructure of $\mathbb{A}$ induced by $N_r(a)$ expanded with a constant for a.

Evaluating a Basic Local Sentence

How do we evaluate a basic local sentence
$\exists x_1 \cdots \exists x_s \left(\bigwedge_{i \neq j} \delta(x_i, x_j) > 2r \wedge \bigwedge_i \psi^r(x_i) \right)$ in a graph $\mathbb{A} \in \mathcal{BD}_k$?

For each $v \in \mathbb{A}$, determine whether

$$\mathrm{Nbd}_r^{\mathbb{A}}(a) \models \psi[a].$$

Since the size of $N_r(a)$ is bounded, this takes linear time.

Label a red if so. We now want to know whether there exists a $2r$-*scattered* set of red vertices of size s.

Finding a Scattered Set

Choose red vertices from $\mathbb{A}$ in some order, removing the $2r$-neighbourhood of each chosen vertex.
$a_1 \in \mathbb{A}$,
$a_2 \in \mathbb{A} \setminus N_{2r}(a_1)$,
$a_3 \in \mathbb{A} \setminus (N_{2r}(a_1) \cup N_{2r}(a_2)), \ldots$

If the process continues for s steps, we have found a $2r$-scattered set of size s.
Otherwise, for some $u < s$ we have found $a_1, \ldots, a_u$ such that all red vertices are contained in

$$N_{2r}(a_1, \ldots, a_u)$$

This is a graph of bounded size and we can determine whether it contains a $2r$-scattered set of *red* vertices by exhaustive search.

Strategy

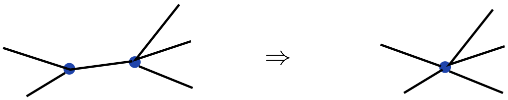
Thus, the overall strategy for evauating a sentence φ in $\mathbb{A}$ is to

1. convert φ to *Gaifman normal form*;
2. for each basic local sentence in the form:
   2.1 evaluate the local formula at every element; and
   2.2 determine whether there is a large enough scattered set among those elements that satisfy it.

Graph Minors

We say that a graph G is a minor of graph H (written $G \preceq H$) if G can be obtained from H by repeated applications of the operations:

- *delete an edge*;
- *delete a vertex* (and all incident edges); and
- *contract an edge*



Graph Minors

A graph $G = (V, E)$ is a *minor* of $H = (U, F)$, if there is a graph $H' = (U', F')$ with $U' \subseteq U$ and $F' \subseteq F$ and a surjective map $M : U' \to V$ such that

- for each $v \in V$, $M^{-1}(v)$ is a *connected subgraph* of H'; and
- for each edge $(u, v) \in E$, there is an edge in F' between some $x \in M^{-1}(u)$ and some $y \in M^{-1}(v)$.

The subgraphs $M^{-1}(v)$ are the *branch sets* witnessing that $G \preceq H$.

We say G *is a minor at depth r of H* ($G \preceq_r H$) if $G \preceq H$ is witnessed by branch sets of radius at most r.

Nowhere-Dense Classes

Definition:
A class of graphs $\mathcal{C}$ is said to be *nowhere dense* if, for each $r \geq 0$ there is a graph H_r such that $H_r \not\preceq_r G$ for any graph $G \in \mathcal{C}$.

This was introduced by Nešetřil and Ossona de Mendez as a formalisation of classes of *sparse* graphs.

We say $\mathcal{C}$ is *effectively nowhere dense* if the function $r \mapsto H_r$ is computable.

Trichotomy

Associate with any infinite class $\mathcal{C}$ of graphs the following parameter:

$$d_{\mathcal{C}} = \lim_{r \to \infty} \limsup_{G \in \mathcal{C}_r} \frac{\log ||G||}{\log |G|},$$

where $\mathcal{C}_r$ is the collection of graphs obtained as minors of a graph in $\mathcal{C}$ by contracting neighbourhoods of radius at most r.

The *trichotomy theorem* states that $d_{\mathcal{C}}$ can only take values 0, 1 and 2.

The nowhere-dense classes are exactly the ones where $d_{\mathcal{C}} \neq 2$.