Homomorphism Preservation on Quasi-Wide Classes

Anuj Dawar
University of Cambridge Computer Laboratory, UK

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Abstract

A class of structures is said to have the homomorphism-preservation property just in case every first-order formula that is preserved by homomorphisms on this class is equivalent to an existential-positive formula. It is known by a result of Rossman that the class of finite structures has this property and by previous work of Atserias et al. that various of its subclasses do. We extend the latter results by introducing the notion of a quasi-wide class and showing that any quasi-wide class that is closed under taking substructures and disjoint unions has the homomorphism-preservation property. We show, in particular, that classes of structures of bounded expansion and that locally exclude minors are quasi-wide. We also construct an example of a class of finite structures which is closed under substructures and disjoint unions but does not admit the homomorphism-preservation property.

1 Introduction

Preservation theorems are model-theoretic results that link semantic restrictions on a logic with corresponding syntactic restrictions. For instance, the Łoś-Tarski preservation theorem guarantees that any first-order formula whose models are closed under extensions is equivalent to an existential formula. In the early development of finite model theory, it was noted that many classical preservation theorems of model theory fail when we are only interested in finite structures (see [11]). The Łoś-Tarski theorem is an example of one such—it was noted by Tait [16] that there are formulas of first-order logic whose *finite* models are closed under extension but that are not equivalent, even in restriction to finite structures, to an existential formula. Similarly, Ajtai and Gurevich [1] established that Lyndon's theorem—which implies that any formula that is monotone on all structures is equivalent to one that is positive—also fails in the finite. One example of a preservation theorem whose status in the finite remained open for many years is the homomorphism preservation theorem. This states that a first-order formula whose models are closed under homomorphisms is equivalent to an existential-positive formula. Rossman recently proved [13] that this holds, even when we restrict ourselves to finite structures.

A recent trend in finite model theory has sought to examine model-theoretic questions, such as the preservation properties, not just on the class of all finite structures but on subclasses that are of interest from the algorithmic point of view (see [5] for an overview of results in this direction). Thus, prior to Rossman's result, Atserias et al. [4] proved that the homomorphism preservation theorem holds in any class of structures $\mathcal{C}$ of *bounded treewidth*, which is closed under substructures and disjoint unions. More generally, they showed that homomorphism preservation holds on $\mathcal{C}$ provided that the *Gaifman graphs* of structures in $\mathcal{C}$ exclude some minor and $\mathcal{C}$ is closed under substructures and disjoint unions. Note that these results are not implied by Rossman's theorem. Indeed, if we consider two classes $\mathcal{C} \subseteq \mathcal{C}'$, we cannot conclude anything about whether or not homomorphism preservation holds on C from the fact that it holds on C'. An example of a class of finite structures on which homomorphism preservation fails is discussed in Section 5.

An open question that was posed in [4] was whether the results from that paper could be extended to other classes, in particular by replacing the requirement that C exclude a minor by the requirement that C have *bounded local treewidth* as defined in [9, 10]. This restriction is incomparable with the requirement that C excludes a minor, in the sense that there are classes with an excluded minor that do not have bounded local treewidth and vice versa. However, there is a common generalisation of the two in the notion of *locally excluded minors* introduced by Dawar et al. [6]. In this paper, we answer the open question from [4] by showing that any class C of finite structures that locally excludes a minor and is closed under taking substructures and disjoint unions satisfies the homomorphism preservation property. We also establish this for classes of *bounded expansion*, as defined by Nešetřil and Ossona de Mendez [14].

The proof given in [4] that classes of structures that exclude a minor satisfy homomorphism preservation was composed of two elements. First, a result derived from a lemma by Ajtai and Gurevich [2] that showed a certain density property for minimal models of a formula φ that is preserved under homomorphisms. This implies that if a class C satisfies the condition of being *almost wide* (this is defined in Section 2 below) and is closed under substructures and disjoint unions, then C satisfies homomorphism preservation. Secondly, we showed, using a combinatorial construction from [12] that any class that excludes some graph as a minor is almost wide. In order to extend these results to classes that locally exclude a minor and classes of bounded expansion, we first define a relaxation of the almost wideness condition to one we term *quasi-wideness*. We show that the Ajtai-Gurevich lemma can be adapted to show that any class C which is quasi-wide and closed under substructures and disjoint unions also satisfies homomorphism preservation. This is established in Section 3. Then, an extension of the combinatorial argument from [4] establishes that classes of bounded expansion and classes that locally exclude a minor are *quasi-wide*. These arguments are presented in Section 4.

The steady recurrence of the requirement that C is closed under substructures and disjoint unions arises from the fact that these are the constructions used in the density argument of Ajtai and Gurevich. A natural question that arises is whether these conditions alone might be sufficient to guarantee homomorphism preservation. However, this is not the case, as we establish through a counter-example constructed in Section 5.

The results presented here were announced (without proof) in an invited lecture [5]. Since then, Nešetřil and Ossona de Mendez have extended the combinatorial argument from Section 4 and provided an elegant characterisation of quasi-wide classes that are closed under substructures [15].

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2 Preliminaries

This section contains the definitions of some basic notions and a minimum amount of background material.

2.1 Relational Structures

A *relational vocabulary* σ is a finite set of *relation symbols*, each with a specified *arity*. A *σ-structure* $\mathbb{A}$ consists of a *universe* A, or *domain*, and an *interpretation* which associates to each relation symbol $R \in \sigma$

of some arity r, a relation $R^{\mathbb{A}} \subseteq A^r$. A *graph* is a structure $\mathbf{G} = (V, E)$, where E is a binary relation that is symmetric and irreflexive. Thus, our graphs are undirected, loopless, and without parallel edges.

A σ-structure $\mathbb{B}$ is called a *substructure* of $\mathbb{A}$ (and we write $\mathbb{B} \subseteq \mathbb{A}$) if $B \subseteq A$ and $R^{\mathbb{B}} \subseteq R^{\mathbb{A}}$ for every $R \in \sigma$. It is called an *induced substructure* if $R^{\mathbb{B}} = R^{\mathbb{A}} \cap B^r$ for every $R \in \sigma$ of arity r. Note that this terminology is at variance with common usage in model theory where the term "substructure" is used for what we call an "induced substructure". However, it is more convenient for us as, for the purpose of studying properties preserved under homomorphisms, we are more interested in substructures that are not necessarily induced. Note also the analogy with the concepts of *subgraph* and *induced subgraph* from graph theory. A substructure $\mathbb{B}$ of $\mathbb{A}$ is proper if $\mathbb{A} \neq \mathbb{B}$.

A *homomorphism* from $\mathbb{A}$ to $\mathbb{B}$ is a mapping $h : A \to B$ from the universe of $\mathbb{A}$ to the universe of $\mathbb{B}$ that preserves the relations, that is if $(a_1, \ldots, a_r) \in R^{\mathbb{A}}$, then $(h(a_1), \ldots, h(a_r)) \in R^{\mathbb{B}}$. We say that two structures $\mathbb{A}$ and $\mathbb{B}$ are *homomorphically equivalent* if there is a homomorphism from $\mathbb{A}$ to $\mathbb{B}$ and a homomorphism from $\mathbb{B}$ to $\mathbb{A}$. Note that, if $\mathbb{A}$ is a substructure of $\mathbb{B}$, then the injection mapping is a homomorphism from $\mathbb{A}$ to $\mathbb{B}$. If the homomorphism h is bijective and its inverse is a homomorphism from $\mathbb{B}$ to $\mathbb{A}$ then $\mathbb{A}$ and $\mathbb{B}$ are isomorphic and we write $\mathbb{A} \cong \mathbb{B}$.

For a pair of structures $\mathbb{A}$ and $\mathbb{B}$, we write $\mathbb{A} \oplus \mathbb{B}$ for the *disjoint union* of $\mathbb{A}$ and $\mathbb{B}$. That is, $\mathbb{A} \oplus \mathbb{B}$ is the structure whose universe is the disjoint union of A and B and where, for any relation symbol R and any tuple of elements $\mathbf{t}$, we have $\mathbf{t} \in R^{\mathbb{A} \oplus \mathbb{B}}$ just in case either $\mathbf{t} \in R^{\mathbb{A}}$ or $\mathbf{t} \in R^{\mathbb{B}}$.

The *Gaifman graph* of a σ-structure $\mathbb{A}$, denoted by $\mathcal{G}(\mathbb{A})$, is the (undirected) graph whose set of nodes is the universe of $\mathbb{A}$, and whose set of edges consists of all pairs (a, a') of distinct elements of A such that a and a' appear together in some tuple of a relation in $\mathbb{A}$.

Let $\mathbf{G} = (V, E)$ be a graph. Moreover, let $u \in V$ be a vertex and let $d \geq 0$ be an integer. The r-*neighborhood* of u in $\mathbf{G}$, denoted by $N_d^{\mathbf{G}}(u)$, is defined inductively as follows:

1. $N_0^{\mathbf{G}}(u) = \{u\}$;
2. $N_{r+1}^{\mathbf{G}}(u) = N_r^{\mathbf{G}}(u) \cup \{v \in V : (v, w) \in E \text{ for some } w \in N_r^{\mathbf{G}}(u)\}$.

Where this causes no confusion, we also write $N_r^{\mathbf{G}}(u)$ for the subgraph of $\mathbf{G}$ induced by this set of vertices. For a structure $\mathbb{A}$ and an element a in its universe, we write $N_r^{\mathbb{A}}(a)$ for the substructure of $\mathbb{A}$ induced by the set $N_r^{\mathcal{G}(\mathbb{A})}(a)$.

2.2 Logic

Let σ be a relational vocabulary. The *atomic formulas* of σ are those of the form $R(x_1, \ldots, x_r)$, where $R \in \sigma$ is a relation symbol of arity r, and $x_1, \ldots, x_r$ are first-order variables that are not necessarily distinct. Formulas of the form $x = y$ are also atomic formulas, and we refer to them as *equalities*. The collection of *first-order formulas* is obtained by closing the atomic formulas under negation, conjunction, disjunction, universal and existential first-order quantification. The semantics of first-order logic is standard. If $\mathbb{A}$ is a σ-structure and φ is a first-order formula, we use the notation $\mathbb{A} \models \varphi[\mathbf{a}]$ to denote the fact that φ is true in $\mathbb{A}$ when its free variables are interpreted by the tuple of elements $\mathbf{a}$. When φ is a sentence (i.e. contains no free variables), we simply write $\mathbb{A} \models \varphi$. The collection of *existential-positive* first-order formulas is obtained by closing the atomic formulas under conjunction, disjunction, and existential quantification. By substituting variables, it is easy to see that equalities can be eliminated from existential-positive formulas.

We say that a first-order formula φ is *preserved under homomorphisms* if, whenever $\mathbb{A} \models \varphi[\mathbf{a}]$ and $h : A \to B$ is a homomorphism from $\mathbb{A}$ to $\mathbb{B}$ then $\mathbb{B} \models \varphi[h(\mathbf{a})]$. It is an easy exercise to show that any existential positive first-order formula is preserved under homomorphisms. The homomorphism preservation theorem provides a kind of converse to this statement: every first-order formula that is preserved under homomorphisms is logically equivalent to an existential positive formula.

We are interested in versions of homomorphism preservation on restricted classes of structures. If C is a class of structures, we say that a formula φ is *preserved under homomorphisms on* C if whenever $\mathbb{A}$ and $\mathbb{B}$ are structures in C, $\mathbb{A} \models \varphi[\mathbf{a}]$ and $h : A \to B$ is a homomorphism from $\mathbb{A}$ to $\mathbb{B}$ then $\mathbb{B} \models \varphi[h(\mathbf{a})]$. We say that two formulas φ and ψ are *equivalent on* C if every structure $\mathbb{A}$ in C verifies $\mathbb{A} \models (\varphi \leftrightarrow \psi)$. We say that C has the *homomorphism preservation property* if every formula φ that is preserved under homomorphisms on C is equivalent on C to an existential positive formula. By a theorem of Rossman [13], the class of finite structures has the homomorphism preservation property.

For a sentence φ preserved under homomorphisms on a class of structures C, we say that $\mathbb{A} \in C$ is a *minimal model* of φ in C if $\mathbb{A} \models \varphi$ and for every substructure $\mathbb{B} \subseteq \mathbb{A}$ such that $\mathbb{B} \in C$, $\mathbb{B} \not\models \varphi$. The following lemma is established by an easy argument sketched in [4].

Lemma 1. *Let C be a class of finite structures closed under taking substructures and let φ be a sentence that is preserved under homomorphisms on C. Then the following are equivalent:*

1. *φ has finitely many minimal models in C.*
2. *φ is equivalent on C to an existential-positive sentence.*

The main consequence of this lemma is that in order to establish that C has the homomorphism preservation property, it suffices to establish an upper bound on the size of the minimal models. To be precise, we aim to prove that for any φ there is an N such that no minimal model of φ is larger than N.

The quantifier rank of a first-order formula φ is just the maximal depth of nesting of quantifiers in φ. For every integer $r \geq 0$, let $\delta(x, y) \leq r$ denote the first-order formula expressing that the distance between x and y in the Gaifman graph is at most r. Let $\delta(x, y) > r$ denote the negation of this formula. Note that the quantifier rank of $\delta(x, y) \leq r$ is bounded by r. A *basic local sentence* is a sentence of the form

$$\exists x_1 \cdots \exists x_n \left(\bigwedge_{i \neq j} \delta(x_i, x_j) > 2r \wedge \bigwedge_i \psi^{N_r(x_i)}(x_i) \right), \tag{1}$$

where ψ is a first-order formula with one free variable. Here, $\psi^{N_r(x_i)}(x_i)$ stands for the relativization of ψ to $N_r(x_i)$; that is, the subformulas of ψ of the form $\exists x\theta$ are replaced by $\exists x(\delta(x, x_i) \leq r \wedge \theta)$, and the subformulas of the form $\forall x\theta$ are replaced by $\forall x(\delta(x, x_i) \leq r \to \theta)$. The *locality radius* of a basic local sentence is r. Its *width* is n. The formula ψ is called the *local condition*.

The main value of basic local sentences is that they form a building block for first-order logic. This follows from Gaifman's Theorem (for a proof, see, for example, [8, Theorem 2.5.1]), which states that every first-order sentence is equivalent to a Boolean combination of basic local sentences. We will need a refined version of this, which takes account of quantifier rank. The following statement follows immediately from the proof given in [8].

Theorem 2 (Gaifman)**.** *Every first-order sentence φ of quantifier rank at most q is equivalent to a Boolean combination of basic local sentences of locality radius at most 7^q.*

Note, in particular, that the upper bound on the locality radius does not depend on the signature σ.

2.3 Graphs

We are interested in classes of finite structures C defined by a graph-theoretic restriction on their Gaifman graphs. In order to define these restrictions, we introduce some graph theoretic concepts. For further details on graph minors, the reader is referred to [7]. For a graph $\mathbf{G}$, we often write $V^{\mathbf{G}}$ for the set of its vertices and $E^{\mathbf{G}}$ for the set of its edges. For $A \subseteq V^{\mathbf{G}}$, we write $\mathbf{G}[A]$ to denote the subgraph of $\mathbf{G}$ induced by the set of vertices A.

We say that a graph $\mathbf{G}$ is a *minor* of $\mathbf{H}$ (written $\mathbf{G} \preceq \mathbf{H}$) if $\mathbf{G}$ can be obtained from a subgraph of $\mathbf{H}$ by contracting edges. The contraction of an edge (u, v) consists in replacing its two endpoints with a new vertex w whose neighbours are all nodes that were neighbours of either u or v. An equivalent characterization (see [7]) states that $\mathbf{G}$ is a minor of $\mathbf{H}$ if there is a map that associates to each vertex v of $\mathbf{G}$ a non-empty *connected* subgraph $\mathbf{H}_v$ of $\mathbf{H}$ such that $\mathbf{H}_u$ and $\mathbf{H}_v$ are disjoint for $u \neq v$ and if there is an edge between u and v in $\mathbf{G}$ then there is an edge in $\mathbf{H}$ between some node in $\mathbf{H}_u$ and some node in $\mathbf{H}_v$. The subgraphs $\mathbf{H}_v$ are called *branch sets*.

We say that a class C of finite graphs *excludes* $\mathbf{G}$ *as a minor* if, for every $\mathbf{H}$ in C, $\mathbf{G} \npreceq \mathbf{H}$. We say that C *excludes a minor* if there is some graph $\mathbf{G}$ such that C excludes $\mathbf{G}$ as a minor. Note that if $\mathbf{G}$ is a graph on n vertices and $\mathbf{K}_n$ is the clique on n vertices, then $\mathbf{G} \preceq \mathbf{K}_n$. Thus, if C excludes a minor, then there is an n such that C excludes $\mathbf{K}_n$ as a minor. Among classes of graphs that exclude a minor are the class of planar graphs, or more generally, the class of graphs embeddable into any given fixed surface.

The notion of graph classes with locally excluded minors is introduced in [6]. We say that a class C *locally excludes minors* if there is a function $f : \mathbb{N} \to \mathbb{N}$ such that for each $\mathbf{G}$ in C and each vertex v in $\mathbf{G}$, $\mathbf{K}_{f(r)} \npreceq N_r^{\mathbf{G}}(v)$. That is, for every r, the class of graphs C_r, formed from C by taking the neighbourhoods of radius r around all vertices of graphs in C, excludes a minor.

Finally, we define classes of bounded expansion, as introduced by Nešetřil and Ossona de Mendez [14]. Suppose a graph $\mathbf{G}$ is a minor of $\mathbf{H}$ as witnessed by the collection of branch sets $\{\mathbf{H}_v \mid v \in V^{\mathbf{G}}\}$. We say that $\mathbf{G}$ is a *minor at depth* r of $\mathbf{H}$ (and write $\mathbf{G} \preceq_r \mathbf{H}$) if each of these branch sets is contained in a neighbourhood of $\mathbf{H}$ of radius r. That is, for each $v \in V^{\mathbf{G}}$, there is a $w \in V^{\mathbf{H}}$ such that $\mathbf{H}_v \subseteq N_r^{\mathbf{H}}(w)$. For any graph $\mathbf{H}$, the *greatest reduced average density* (or *grad*) *of radius* r of $\mathbf{H}$, written $\nabla_r(\mathbf{H})$ is defined as

$$\nabla_r(\mathbf{H}) = \max\{\frac{|E^{\mathbf{G}}|}{|V^{\mathbf{G}}|} \mid \mathbf{G} \preceq_r \mathbf{H}\}.$$

In other words, $\nabla_r(\mathbf{H})$ is half the maximum average degree that occurs among minors of $\mathbf{H}$ of depth r. In particular, if $d(\mathbf{G})$ denotes the average degree of $\mathbf{G}$, then $\nabla_0(\mathbf{H}) = \max\{\frac{1}{2}d(\mathbf{G}) \mid \mathbf{G} \subseteq \mathbf{H}\}$.

A class of graphs C is said to be of *bounded expansion* if there is a function $f : \mathbb{N} \to \mathbb{N}$ such that for every graph $\mathbf{G}$ in C, $\nabla_r(\mathbf{G}) \leq f(r)$. It is known that for every n, any graph with average degree $10n^2$ contains $\mathbf{K}_n$ as a minor (see [7, Theorem 7.2.1]. It follows immediately that if C excludes $\mathbf{K}_n$ as a minor, it has bounded expansion. Indeed, the constant function $f(r) = 10n^2$ witnesses this.

Any class C that excludes a minor both has bounded expansion and locally excludes minors. However, the lat two restrictions are known to be incomparable in the sense that there are classes C that locally exclude minors but are not of bounded expansion and vice versa (see [6]). Another condition on a class C, considered in [4] is that it has *bounded degree*. That is to say that there is a constant d such that every vertex in every graph in C has degree at most d. This restriction is incomparable with the requirement that C excludes a minor but again, it is immediate that any class of bounded degree both locally excludes minors and has bounded expansion. See [5] for a map of these various conditions and implications between them.

2.4 Homomorphism Preservation Theorems

In [4], the homomorphism preservation property is established for a number of classes of structures, based on certain combinatorial properties that were called *wide* and *almost wide* in [3]. In the following, when we talk of a class of finite structures $\mathcal{C}$ satisfying a graph-theoretic restriction, such as excluding a minor, we mean that the collection of Gaifman graphs $\mathcal{G}(\mathbb{A})$ of structures $\mathbb{A}$ in $\mathcal{C}$ satisfies the condition.

Definition 3. *A set of elements B in a σ-structure $\mathbb{A}$ is r-*scattered *if for every pair of distinct $a, b \in B$ we have $N_r^{\mathbb{A}}(a) \cap N_r^{\mathbb{A}}(b) = \emptyset$.*

We say that a class of finite σ-structures $\mathcal{C}$ is wide *if for every r and m there exists an N such that every structure in $\mathcal{C}$ of size at least N contains an r-scattered set of size m.*

It is easy to see that if $\mathcal{C}$ has bounded degree, then it is wide.

Definition 4. *A class of finite σ-structures $\mathcal{C}$ is* almost wide with margin *k if for every r and m there exists an N such that every structure $\mathbb{A}$ with at least N elements in $\mathcal{C}$ contains a set B with at most k elements such that $\mathcal{G}(\mathbb{A})[A \setminus B]$ contains an r-scattered set of size m.*

We say that $\mathcal{C}$ is almost wide *if there is some k such that it is almost wide with margin k.*

An example is the class of acyclic graphs, which is not wide (as we have arbitrarily large trees where the distance between any two vertices is 2) but is almost wide with margin 1. More generally, it is shown in [4] that if $\mathcal{C}$ excludes $\mathbf{K}_n$ as a minor, then $\mathcal{C}$ is almost wide with margin $n - 2$.

A theorem of [4] shows that almost wideness, along with some natural closure properties of a class $\mathcal{C}$ is sufficient to guarantee the homomorphism preservation property.

Theorem 5 ([4])**.** *Any class $\mathcal{C}$ of finite σ-structures that is almost wide and is closed under taking substructures and disjoint unions of structures has the homomorphism preservation property.*

This is proved using a lemma of Ajtai and Gurevich which we review in Section 3. Thus, as long as $\mathcal{C}$ is closed under substructures and disjoint unions, if it has bounded degree, bounded treewidth or excludes a minor, it has the homomorphism preservation property. An open question posed in [4] was whether the same could be proved in the case where $\mathcal{C}$ has *bounded local treewidth*. We will not define this notion formally here but only note that any class of bounded local treewidth also locally excludes minors. Thus, by establishing the homomorphism preservation property for classes that locally exclude minors, we settle the open question.

3 Quasi-Wide Classes of Structures

By Theorem 5, the homomorphism preservation property holds for classes of structures which are almost wide and closed under taking substructures and homomorphisms. Unfortunately, knowing that a class $\mathcal{C}$ has bounded expansion or that it locally excludes minors is not sufficient to establish that it is almost wide. Our aim in this section is to show that the condition of almost wideness can be relaxed to a weaker condition that is satisfied by the classes we consider. We proceed to define this condition.

Definition 6. *Let $f : \mathbb{N} \to \mathbb{N}$ be a function. A class of finite σ-structures $\mathcal{C}$ is* quasi-wide with margin *f if for every r and m there exists an N such that every structure $\mathbb{A}$ with at least N elements in $\mathcal{C}$ contains a set B with at most $f(r)$ elements such that $\mathcal{G}(\mathbb{A})[A \setminus B]$ contains an r-scattered set of size m.*

We say that $\mathcal{C}$ is quasi-wide *if there is some f such that $\mathcal{C}$ is quasi-wide with margin f.*

In other words, unlike in the definition of almost wide classes, the number of elements we need to remove to guarantee a large scattered set in a large enough structure $\mathbb{A}$ can be allowed to depend on the radius r of the neighbourhoods we consider.

Theorem 5 is obtained from the following lemma proved by Ajtai and Gurevich [2] and the observation that the only constructions used in the proof involve taking substructures and disjoint unions. We sketch an outline of the proof later.

Lemma 7 (Ajtai-Gurevich)**.** *For any sentence φ that is preserved under homomorphisms and any $k \in \mathbb{N}$, there are $r, m \in \mathbb{N}$ such that if $\mathbb{A}$ is a minimal model of φ and $B \subseteq A$ is a set of its elements with $|B| \leq k$, then $\mathcal{G}(\mathbb{A})[A \setminus B]$ does not contain an r-scattered set of size m.*

Our aim here is to show that in the proof of Lemma 7, the value of r can be chosen independently of the value of k. This will immediately allow us to extend Theorem 5 to quasi-wide classes of structures. We proceed with an outline of the proof of Ajtai and Gurevich.

The first step in the proof is to prove it for the case when $k = 0$. Then, the general case is reduced to this special case. So, suppose φ is a sentence of quantifier rank q that is preserved under homomorphisms. Let $\Sigma = \{\varphi_1, \ldots, \varphi_s\}$ be a collection of basic local sentences (obtained by Theorem 2) such that φ is equivalent to a Boolean combination of them. For each i, let t_i be the radius of locality, n_i the width and $\psi_i(x)$ the local condition of φ_i. Also let $t = \max_i t_i$ and $n = \max_i n_i$. We take $r = 2t$ and $m = 2^s + 1$. For each i, we write $\theta_i(y)$ for the following formula

$$\exists x(\delta(x, y) \leq t_i \wedge \psi_i^{N_{t_i}(x)}(x)).$$

Suppose then that $\mathbb{A}$ is a model of φ that contains an r-scattered set of size m. We wish to show that $\mathbb{A}$ cannot be minimal. Suppose that $C = \{c_1, \ldots, c_m\}$ is the r-scattered set. Then, by definition $N_r^{\mathbb{A}}(c_i) \cap N_r^{\mathbb{A}}(c_j) = \emptyset$ for $i \neq j$. Furthermore, since $m > s$, there are i and j with $i \neq j$ such that for all l, $\mathbb{A} \models \theta_l[c_i]$ if, and only if, $\mathbb{A} \models \theta_l[c_j]$. Let $\mathbb{B}$ be the substructure of $\mathbb{A}$ obtained by removing some tuple that includes c_i from some relation R of $\mathbb{A}$ (if there is no such relation, then we can get a model of φ by removing the element c_i, showing that $\mathbb{A}$ is not minimal in any case). Finally, we take $\mathbb{B}_n$ to be the structure that is the disjoint union of n copies of $\mathbb{B}$ and $\mathbb{A}_n$ to be the structure that is the disjoint union of $\mathbb{A}$ and $\mathbb{B}_n$. Ajtai and Gurevich prove that the structures $\mathbb{A}_n$ and $\mathbb{B}_n$ must agree on the sentence φ. Since φ is preserved under homomorphisms, and there are homomorphisms from $\mathbb{A}$ to $\mathbb{A}_n$ and from $\mathbb{B}_n$ to $\mathbb{B}$, it follows that if $\mathbb{A}$ is a model of φ so is $\mathbb{B}$. Thus, since $\mathbb{B}$ is a substructure of $\mathbb{A}$, the latter is not a minimal model of φ.

Note that, if C is a class of structures that is closed under substructures and disjoint unions then, whenever it contains $\mathbb{A}$, it also contains $\mathbb{B}$, $\mathbb{B}_n$ and $\mathbb{A}_n$. Thus the above argument showing that $\mathbb{A}$ is not minimal works in restriction to such a class. Note further that in the above argument establishing Lemma 7 for $k = 0$, the values of r and m depend on φ, but r can be bounded above by $2 \cdot 7^q$ where q is the quantifier rank of φ, *independently of the signature* σ. A similar upper bound for m is not obtained as this depends on the number of inequivalent basic local sentences of a given quantifier rank and locality radius that can be expressed and this, in turn, depends on the signature.

The proof of Lemma 7 by Ajtai and Gurevich then proceeds to reduce the case $k > 0$ to the case $k = 0$ by means of the construction of what they call *plebian companions*. That is, for every structure $\mathbb{A}$ and a tuple of elements $\mathbf{a} = (a_1, \ldots, a_k)$ from $\mathbb{A}$ we define a structure $p\mathbb{A}_{\mathbf{a}}$ called the *plebian companion* of $\mathbb{A}$. This is a structure over a richer vocabulary than $\mathbb{A}$ and has the property that $\mathcal{G}(p\mathbb{A}_{\mathbf{a}}) \cong \mathcal{G}(\mathbb{A})[A \setminus \mathbf{a}]$. In particular, $p\mathbb{A}_{\mathbf{a}}$ contains an r-scattered set of m elements if, and only if, removing the elements $a_1, \ldots, a_k$ from $\mathbb{A}$ creates such a set. Furthermore, Ajtai and Gurevich give a translation that takes a formula φ in the signature τ of $\mathbb{A}$ to a formula $\widehat{\varphi}$ in the signature τ' of $p\mathbb{A}_{\mathbf{a}}$ so that $\mathbb{A} \models \varphi$ if, and only if, $p\mathbb{A}_{\mathbf{a}} \models \widehat{\varphi}$ and $\widehat{\varphi}$ is preserved under homomorphisms if φ is. This then allows us to deduce Lemma 7 since if $\mathbb{A}$ is a model of φ

and $B = \{a_1, \ldots, a_k\}$ a set of elements such that $\mathcal{G}(\mathbb{A})[A \setminus B]$ contains an r-scattered set of m elements, we can note (from the case $k = 0$) that $p\mathbb{A}_{\mathbf{a}}$ is not a minimal model of $\widehat{\varphi}$. Moreover, from a submodel of the latter we can reconstruct a proper substructure of $\mathbb{A}$ that is a model of φ establishing that $\mathbb{A}$ is not minimal.

Our aim here is to observe that in the translation of φ to $\widehat{\varphi}$ while the signature of $\widehat{\varphi}$ depends on the value of k, the quantifier rank is actually the same as that of φ. To this end, we give the translation in detail.

Fix a structure $\mathbb{A}$ in a relational signature τ and a tuple of elements $a_1, \ldots, a_k$ from A. The signature τ' contains all the relation symbols in τ. In addition, for each relation symbol R of arity r in τ and each non-empty partial function $m : \{1, \ldots, r\} \rightharpoonup \{a_1, \ldots, a_k\}$, τ' contains a new relation symbol R_m whose arity is $r - j$ where j is the number of elements of $\{1, \ldots, r\}$ on which m is defined. In particular, if m is total, $r = j$ and R_m is then a 0-ary relation symbol. That is to say, it is a Boolean symbol that is interpreted as either true or false in any τ'-structure.

The universe of $p\mathbb{A}_{\mathbf{a}}$ is obtained from that of $\mathbb{A}$ by excluding the elements $a_1, \ldots, a_k$. For each relation symbol R in τ, the interpretation of R in $p\mathbb{A}_{\mathbf{a}}$ is the restriction of $R^{\mathbb{A}}$ to the universe of $p\mathbb{A}_{\mathbf{a}}$. To define the interpretation of R_m, let $\mathbf{b}$ be an $r - j$ tuple of elements from $p\mathbb{A}_{\mathbf{a}}$. Let $\mathbf{b}'$ be the r-tuple of elements of $\mathbf{A}$ obtained from $\mathbf{b}$ by inserting in position i the element $m(i)$. We say that $\mathbf{b} \in R_m^{p\mathbb{A}_{\mathbf{a}}}$ if, and only if, $\mathbf{b}' \in R^{\mathbb{A}}$. In the special case that R_m is 0-ary, we say that it is interpreted as true if, and only if, the unique empty tuple is in R_m by the above rule.

To describe the translation of φ to $\widehat{\varphi}$, we consider an expansion of the signature τ with constants for the elements $a_1, \ldots, a_k$ (we do not distinguish between the elements and the constants that name them). Note that these constants appear neither in φ nor in $\widehat{\varphi}$ but they are useful in the inductive definition of the translation. So we proceed to define the translation by induction on the structure of a formula φ in the expanded signature.

- If φ is the atomic formula $R\mathbf{t}$ and the tuple of terms $\mathbf{t}$ does not contain any of the constants $a_1, \ldots, a_k$, then $\widehat{\varphi} := \varphi$.

- If φ is the atomic formula $R\mathbf{t}$ and $\mathbf{t}$ contains constants from $a_1, \ldots, a_k$, let m be the partial function that takes i to the constant appearing in position i of $\mathbf{t}$. Also, let $\mathbf{t}'$ be the tuple of variables obtained from $\mathbf{t}$ by removing the constants. Then $\widehat{\varphi} := R_m\mathbf{t}'$.

- If φ is $\neg\psi$, then $\widehat{\varphi} := \neg\widehat{\psi}$ and if φ is $\psi_1 \wedge \psi_2$ then $\widehat{\varphi} := \widehat{\psi_1} \wedge \widehat{\psi_2}$.

- If φ is $\exists x\psi$ then $\widehat{\varphi} := \exists x\widehat{\psi} \vee \bigvee_{i=1}^{k} \widehat{\psi[x/a_i]}$.

It is clear from this translation that, while the signature of $\widehat{\varphi}$ depends on the value of k, its quantifier rank is the same as the quantifier rank of φ. Combining this with the fact that in the proof of Lemma 7 for the case $k = 0$, we could bound the value of r by $2 \cdot 7^q$ independently of the signature of φ, gives us the following strengthening of Lemma 7.

Lemma 8. *For any sentence φ of quantifier rank q that is preserved under homomorphisms and any $k \in \mathbb{N}$, there is an $m \in \mathbb{N}$ such that if $\mathbb{A}$ is a minimal model of φ and $B \subseteq A$ is a set of its elements with $|B| \leq k$, then $\mathcal{G}(\mathbb{A})[A \setminus B]$ does not contain a $2 \cdot 7^q$-scattered set of size m.*

Since, by the observation in [4], this holds relativised to any class of structures $\mathcal{C}$ closed under substructures and disjoint unions, we obtain the following theorem.

Theorem 9. *Any class $\mathcal{C}$ of structures that is quasi-wide and closed under substructures and disjoint unions has the homomorphism preservation property.*

Proof. Let $f : \mathbb{N} \to \mathbb{N}$ be such that C is quasi-wide with margin f. Let φ be a sentence that is preserved under homomorphisms on C. By Lemma 1 it suffices to prove that there is an N such that no minimal model of φ in C has more than N elements.

Write q for the quantifier rank of φ, let $r := 2 \cdot 7^q$ and let $k := f(r)$. Lemma 8 then gives us an m such that in any minimal model of φ the removal of k elements cannot create an r-scattered set of size m. However, Definition 6 ensures that there is an N such that any structure in C with more than N elements contains k elements whose removal creates just such a scattered set. We conclude that no minimal model of φ contains more than N elements. □

4 Bounded Expansion and Locally Excluded Minors

Our aim in this section is to show that classes of graphs that locally exclude minors or that have bounded expansion are quasi-wide. The proof of this is an adaptation of the proof from [4] that classes of structures that exclude a minor are almost wide. To be precise, it is shown there that the following holds.

Theorem 10 ([4])**.** *For any $k, r, m \in \mathbb{N}$ there is an $N \in \mathbb{N}$ such that if $\mathbf{G} = (V, E)$ is a graph with more than N vertices then*

1. *either $\mathbf{K}_k \preceq \mathbf{G}$; or*
2. *there is a set $B \subseteq V$ with $|B| \leq k-2$ such that $\mathbf{G}[V \setminus B]$ contains an r-scattered set of size m.*

The proof of Theorem 10 is a Ramsey-theoretic argument that proceeds by starting with a set of $S \subseteq V$ with N elements and constructing two sequences of sets: $S =: S_0 \supseteq S_1 \supseteq \cdots \supseteq S_r$ and $\emptyset =: B_0 \subseteq B_1 \subseteq \cdots \subseteq B_r$ such that for each $x, y \in S_i$ we have $N_i^{\mathbf{G}[V \setminus B_i]}(x) \cap N_i^{\mathbf{G}[V \setminus B_i]}(y) = \emptyset$. If $\mathbf{K}_k \not\preceq \mathbf{G}$ then we can carry the construction through for r stages and $|S_r| \geq m$ and $|B_r| \leq k-2$. If the construction fails at some stage $i \leq r$, it is because we have found that $\mathbf{K}_k$ is a minor of $\mathbf{G}$ and this can happen in one of three ways.

- We find that there are $s_1, \ldots, s_k \in S_i$ such that for each $1 \leq j < l \leq k$, there is an edge between some vertex in $N_i^{\mathbf{G}[V \setminus B_i]}(s_j)$ and $N_i^{\mathbf{G}[V \setminus B_i]}(s_l)$. In this case, we can take the collection of sets $N_i^{\mathbf{G}[V \setminus B_i]}(s_j)$ for $1 \leq j \leq k$ as branch sets.
- We find that there are $s_1, \ldots, s_k \in S_i$ such that there are distinct vertices x_{jl} for each $1 \leq j < l \leq k$, where each x_{jl} is a neighbour to some vertex in $N_i^{\mathbf{G}[V \setminus B_i]}(s_j)$ and to some vertex in $N_i^{\mathbf{G}[V \setminus B_i]}(s_l)$. In this case, we find that $\mathbf{K}_k$ is a minor of $\mathbf{G}$ by taking as branch sets $N_i^{\mathbf{G}[V \setminus B_i]}(s_j) \cup \{x_{jl} \mid j < l\}$ for $1 \leq j \leq k$.
- We find $s_1, \ldots, s_{k-1} \in S_i$ and vertices $x_1, \ldots, x_{k-1}$ such that x_j has edges connecting it to each of the sets $N_i^{\mathbf{G}[V \setminus B_i]}(s_j)$. Thus, $\mathbf{K}_k$ is found as a minor of $\mathbf{G}$ by taking as branch sets: $N_i^{\mathbf{G}[V \setminus B_i]}(s_j) \cup \{x_j\}$ for $1 \leq j \leq k-2$ along with $N_i^{\mathbf{G}[V \setminus B_i]}(s_{k-1})$ and $\{x_{k-1}\}$.

The point of this brief recapitulation of the proof is to note that when $\mathbf{K}_k$ is found as a minor of $\mathbf{G}$ in case (1) of the theorem, the branch sets have radius at most $r+1$. Thus, we actually obtains the following stronger theorem.

Theorem 11. *For any $k, r, m \in \mathbb{N}$ there is an $N \in \mathbb{N}$ such that if $\mathbf{G} = (V, E)$ is a graph with more than N vertices then*

1. *either $\mathbf{K}_k \preceq_{r+1} \mathbf{G}$; or*

2. *there is a set* $B \subseteq V$ *with* $|B| \leq k-2$ *such that* $\mathbf{G}[V \setminus B]$ *contains an* r*-scattered set of size* m*.*

We write $N(k,r,m)$ for the value of N obtained from Theorem 11 for given k,r and m.
The following result now follows immediately.

Theorem 12. *Any class of graphs of bounded expansion is quasi-wide.*

Proof. Suppose that C is a class of graphs of bounded expansion and let f be a function such that for any graph $\mathbf{G}$ in C, $\nabla_r(\mathbf{G}) \leq f(r)$. Let $k(r) := 2f(r+1)+2$. Note that

$$\frac{|E^{\mathbf{K}_{k(r)}}|}{|V^{\mathbf{K}_{k(r)}}|} = \frac{k(r)-1}{2} > f(r+1)$$

and therefore, by the definition of bounded expansion, $\mathbf{K}_{k(r)} \not\preceq_{r+1} \mathbf{G}$ for any graph $\mathbf{G}$ in C. Thus, by Theorem 11, if $\mathbf{G}$ has more than $N(k(r),r,m)$ vertices, it contains a set B with at most $k(r)-2$ vertices such that $\mathbf{G}[V^{\mathbf{G}} \setminus B]$ contains an r-scattered set of size m. Thus, C is quasi-wide with margin $k(r)-2$. □

We now consider the case of classes with locally excluded minors. It is useful to first derive a straightforward corollary to Theorem 11.

Corollary 13. *If* $\mathbf{G} = (V,E)$ *is a graph with more than* $N(k,r,m)$ *vertices then*

1. *either there is a* $v \in V$ *such that* $\mathbf{K}_k \preceq N^{\mathbf{G}}_{3r+4}(v)$*; or*

2. *there is a set* $B \subseteq V$ *with* $|B| \leq k-2$ *such that* $\mathbf{G}[V \setminus B]$ *contains an* r*-scattered set of size* m*.*

Proof. Suppose condition (2) fails. Then, by Theorem 11, we have $\mathbf{K}_k \preceq_{r+1} \mathbf{G}$. Let $\mathbf{H}_1,\dots,\mathbf{H}_k$ be the branch sets that witness this and let $v_1,\dots,v_k$ be vertices such that $\mathbf{H}_i \subseteq N^{\mathbf{G}}_{r+1}(v_i)$. Then, for any j and any vertex u in $\mathbf{H}_j$ there is a path of length at most $3r+4$ from v_i to u. This is because there is an edge between some vertex w in $\mathbf{H}_i$ and a vertex w' in $\mathbf{H}_j$. Moreover, there is a path of length at most $r+1$ from v_i to w and since $u, w' \in N^{\mathbf{G}}_{r+1}(v_j)$, there is a path of length at most $2r+2$ from w' to u. Thus, $\bigcup_{j=1}^{k} \mathbf{H}_j \subseteq N^{\mathbf{G}}_{3r+4}(v_i)$ and hence $\mathbf{K}_k \preceq N^{\mathbf{G}}_{3r+4}(v_i)$. □

Theorem 14. *Any class of graphs that locally excludes minors is quasi-wide.*

Proof. Suppose C is a class of graphs that locally excludes minors. In particular, let f be a function such that for any r, $\mathbf{K}_{f(r)} \not\preceq N^{\mathbf{G}}_r(v)$ for any graph $\mathbf{G}$ in C and any vertex v of $\mathbf{G}$.

Now, for any r, let $k(r) := f(3r+4)$. By definition, for any graph $\mathbf{G}$ in C and any vertex v of $\mathbf{G}$, $K_{k(r)} \not\preceq N^{\mathbf{G}}_{3r+4}(v)$. Thus, by Corollary 13, if $\mathbf{G}$ has more than $N(k(r),r,m)$ vertices, it contains a set B with at most $k(r)-2$ vertices such that $\mathbf{G}[V^{\mathbf{G}} \setminus B]$ contains an r-scattered set of size m. Thus, C is quasi-wide with margin $k(r)-2$. □

We can now state the main results of the paper.

Theorem 15. *Any class* C *of finite structures that has bounded expansion and is closed under taking substructures and disjoint unions has the homomorphism preservation property.*

Proof. Immediate from Theorem 9 and Theorem 12. □

Theorem 16. *Any class* C *of finite structures that locally excludes minors and is closed under taking substructures and disjoint unions has the homomorphism preservation property.*

Proof. Immediate from Theorem 14 and Theorem 12. □

5 Failure of Preservation

In this section we give an example of a class of structures $\mathcal{S}$ which is closed under substructures and disjoint unions but does not have the homomorphism preservation property.

The class $\mathcal{S}$ is over a signature τ with two binary relations O and S and one unary relation P. For any $n \in \mathbb{N}$, let L_n be the τ-structure over the universe $\{1, \ldots, n\}$ in which O is interpreted as the usual linear order, i.e. $O(i, j)$ just in case $i < j$; S is the successor relation: $S(i, j)$ just in case $j = i + 1$; and P is interpreted by the set $\{1, n\}$ containing the two endpoints. Let $\mathcal{L}$ be the class of structures isomorphic to L_n for some n. Then $\mathcal{S}$ is the closure of $\mathcal{L}$ under substructures and disjoint unions. Note that every structure $\mathbb{A}$ in $\mathcal{S}$ is the disjoint union of a collection $A_1, \ldots, A_s$ of structures, each of which is a substructure of some L_n.

We begin with some observations about structures in $\mathcal{S}$.

Lemma 17. *If $\mathbb{A}$ is a structure such that $\mathbb{A} \subseteq L_m$ for some m and there is a homomorphism $h : L_n \to \mathbb{A}$ for some $n \geq 2$, then $L_n \cong \mathbb{A}$.*

Proof. Note that, by definition of the structures L_m, if $O(a, b)$ for two elements a, b of $\mathbb{A}$ then $a \neq b$. Since L_n contains two elements $1, n$ in the set P with $O(1, n)$ we conclude that $\mathbb{A}$ contains both endpoints of L_m and they are both in the set $P^{\mathbb{A}}$. Furthermore, L_n contains an S-path from 1 to n. The image of this path under h must be an S-path between the end points of L_m and we conclude that $m = n$ and h is the identity map. Finally, suppose that for some i, j in L_m with $i < j$, the pair (i, j) is not in $O^{\mathbb{A}}$. But then, since $(i, j) \in O^{L_n}$ and h is the identity, h is not a homomorphism. We conclude that $\mathbb{A} \cong L_n$. □

Say that a structure $\mathbb{A} \in \mathcal{S}$ *contains a complete order* if there is some $n \geq 2$ such that $L_n \subseteq \mathbb{A}$.

Lemma 18. *If $\mathbb{A}$ and $\mathbb{B}$ in $\mathcal{S}$ are such that $\mathbb{A}$ contains a complete order and there is a homomorphism $h : \mathbb{A} \to \mathbb{B}$, then $\mathbb{B}$ contains a complete order.*

Proof. Suppose $L_n \subseteq \mathbb{A}$ and $\mathbb{B} = B_1 \oplus \cdots \oplus B_s$ where for each i, $B_i \subseteq L_m$ for some m. Since the B_i are pairwise disjoint and L_n is connected there is some i such that $h(L_n) \subseteq B_i$. But then, by Lemma 17, $B_i \cong L_n$ and so $\mathbb{B}$ contains a complete order. □

Our aim now is to construct a first-order sentence that defines those structures in $\mathcal{S}$ that contain a complete order.

We write $x \leq y$ as an abbreviation for the formula $O(x, y) \vee x = y$. Let $\beta(x, y, z)$ denote the formula $x \leq z \wedge z \leq y$ and let $\lambda(x, y)$ denote the formula that asserts that $O(x, y)$ and that $\leq$ linearly orders the set of elements $\{z \mid x \leq z \text{ and } z \leq y\}$. That is, $\lambda(x, y)$ is the formula:

$$O(x, y) \wedge \forall z_1 \forall z_2 (\beta(x, y, z_1) \wedge \beta(x, y, z_2)) \to (z_1 \leq z_2 \vee z_2 \leq z_1).$$

Let $\nu(z_1, z_2)$ denote the formula $O(z_1, z_2) \wedge \forall w \neg (O(z_1, w) \wedge O(w, z_2))$. In words, $\nu(z_1, z_2)$ defines the pairs of elements in the relation O with nothing in between them. We are now ready to define the sentence φ:

$$\begin{aligned} &\exists x \exists y (P(x) \wedge P(y) \wedge \lambda(x, y) \wedge \\ &\qquad \wedge \forall z_2 \forall z_2 (\beta(x, y, z_1) \wedge \beta(x, y, z_2) \wedge \nu(z_1, z_2)) \to S(z_1, z_2)). \end{aligned}$$

That is, φ asserts that there exist two elements x and y in the relation P such that the set $\{z \mid x \leq z \text{ and } z \leq y\}$ is linearly ordered by O and any two successive elements in that linear order are related by S.

Lemma 19. *For any $\mathbb{A}$ in $\mathcal{S}$, $\mathbb{A} \models \varphi$ if, and only if, $\mathbb{A}$ contains a complete order.*

Proof. It is clear that if $L_n \subseteq \mathbb{A}$, then $\mathbb{A} \models \varphi$ with the endpoints of L_n being witnesses to the outer existential quantifiers. For the converse, suppose that $\mathbb{A} \models \varphi$ and a and b are elements witnessing the outer existential quantifiers. By the facts $P(a)$, $P(b)$ and $O(a,b)$ we know that there is an $A_i \subseteq \mathbb{A}$ and an n such that $A_i \subseteq L_n$ with a, b being the endpoints of L_n. The sentence φ then guarantees that A_i contains all elements of L_n and all tuples in the relations. Thus $A_i \cong L_n$ and so $\mathbb{A}$ contains a complete order. □

Lemma 20. *The formula φ is preserved under homomorphisms on the class $\mathcal{S}$.*

Proof. Immediate from Lemmas 18 and 19. □

Lemma 21. *There is no existential positive formula equivalent to φ on $\mathcal{S}$.*

Proof. By Lemma 1, it suffices to show that φ has infinitely many minimal models in $\mathcal{S}$. But this is immediate as for every $n \geq 2$, L_n is a model of φ but no substructure of L_n is a model of φ. □

It is worth remarking that the collection of Gaifman graphs of structures in $\mathcal{S}$ is the class of all graphs and hence is certainly not quasi-wide.

6 Conclusions

When C is a class of finite structures, there are essentially two methods known for showing that it has the homomorphism preservation property. One is the method used by Rossman to establish the property for the class of all finite structures, based on constructing sufficiently saturated structures. This method works on any class closed under co-retracts. The other, quite distinct method, developed by Atserias et al., is based on the density of minimal models and works for classes of sparse structures, i.e. classes in which any sufficiently large structure is guaranteed not to be dense. In the present paper, we have pushed the latter method further and established the homomorphism preservation property for a richer collection of classes. None of these classes, it appears, is closed under the kind of saturation construction used by Rossman and therefore those methods would not apply.

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