1. A second-order Horn sentence (SO-Horn sentence, for short) is one of the form

\[ Q_1 R_1 \ldots Q_p R_p (\forall x \bigwedge \bigcup C_i) \]

where, each \( Q_i \) is either \( \exists \) or \( \forall \), each \( R_i \) is a relational variable and each \( C_i \) is a Horn clause, which is defined for our purposes as a disjunction of atomic and negated atomic formulae such that it contains at most one positive occurrence of a relational variable. A sentence is said to be ESO-Horn if it is as above, and all \( Q_i \) are \( \exists \).

(a) Show that any ESO-Horn sentence in a relational signature defines a class of structures decidable in polynomial time.

(b) Show that, if \( K \) is an isomorphism-closed class of structures in a relational signature including \( < \), such that each structure in \( K \) interprets \( < \) as a linear order and

\[ \{ [A]_\prec \mid A \in K \} \]

is decidable in polynomial time, then there is a ESO-Horn sentence that defines \( K \).

(c) Show that any SO-Horn sentence is equivalent to a ESO-Horn sentence.

2. The directed graph reachability problem is the problem of deciding, given a structure \((V, E, s, t)\) where \( E \) is an arbitrary binary relation on \( V \), and \( s, t \in V \), whether \( (s, t) \) is in the reflexive-transitive closure of \( E \). This problem is known to be decidable in NL.

Transitive closure logic is the extension of first-order logic with an operator \( \text{tc} \) which allows us to form formulae

\[ \phi \equiv [\text{tc}_{x,y} \psi](t_1, t_2) \]

where \( x \) and \( y \) are \( k \)-tuples of variables and \( t_1 \) and \( t_2 \) are \( k \)-tuples of terms, for some \( k \); and all occurrences of variables \( x \) and \( y \) in \( \psi \) are bound in \( \phi \). The semantics is given by saying, if \( a \) is an interpretation for the free variables of \( \phi \), then \( A \models \phi[a] \) just in case \( (t_1^a, t_2^a) \) is in the reflexive-transitive closure of the binary relation defined by \( \psi(x, y) \) on \( A^k \).

(a) Show that any class of structures definable by a sentence \( \phi \), as above, where \( \psi \) is first-order, is decidable in NL.
(b) Show that, if $K$ is an isomorphism-closed class of structures in a relational signature including $<$, such that each structure in $K$ interprets $<$ as a linear order and

$$\{[A]_< \mid A \in K\}$$

is decidable in NL, then there is a sentence of transitive-closure logic that defines $K$.

3. For a binary relation $E$ on a set $A$, define its deterministic transitive closure to be the set of pairs $(a, b)$ for which there are $c_1, \ldots, c_n \in A$ such that $a = c_1$, $b = c_n$ and for each $i < n$, $c_{i+1}$ is the unique element of $A$ with $(c_i, c_{i+1}) \in E$.

Let DTC denote the logic formed by extending first-order logic with an operator $\text{dtc}$ with syntax analogous to $\text{tc}$ above, where $[\text{dtc}_{x,y} \psi]$ defines the deterministic transitive closure of $\psi(x,y)$.

(a) Show that every sentence of DTC defines a class of structures decidable in L.

(b) Show that, if $K$ is an isomorphism-closed class of structures in a relational signature including $<$, such that each structure in $K$ interprets $<$ as a linear order and

$$\{[A]_< \mid A \in K\}$$

is decidable in L, then there is a sentence of DTC that defines $K$.

4. Show that every sentence of PFP defines a class of structures decidable in PSPACE, and that PFP captures PSPACE on ordered structures, in the same sense as above.

5. Suppose $\phi$ is formula of PFP, $R$ is a relational variable, and $O$ is the class of structures that interpret the symbol $<$ as a linear order. Show there is a formula of PFP that is equivalent to $\exists R \phi$ on all structures in $O$. Use this fact to conclude that a class $K$ of structures is definable by a sentence of the form $\exists R \phi$ (where $\phi$ is in PFP) if, and only if, $\{[A]_< \mid A \in K \text{ and } < \text{ is any order on } A\}$ is in PSPACE.

6. For a signature $\sigma$, a canonical labelling function for $\sigma$-structures is a function $l$ on strings such that, if $A$ is a finite $\sigma$-structure and $<$ an order on its universe, then $l([A]_<) = [A]_{<'}$, for some order $<'$; and if $<_1$ and $<_2$ are any orders on the universe of $A$, $l([A]_{<_1}) = l([A]_{<_2})$.

Show that, if there is a polynomial-time computable canonical labelling function for $\sigma$-structures, then the polynomial-time properties of $\sigma$-structures are recursively enumerable.