Mathematical Tripos Part III
Finite Model Theory

Lent 2002 Anuj Dawar

Exercise Sheet 1

1. The first part is not an exercise in *finite* model theory, but a preservation theorem in the classical mould.

   (a) Show that for any theory T, and sentence ϕ, ϕ is equivalent modulo T to a sentence of L^k if, and only if, for $\mathcal{A}, \mathcal{B} \models T$, with $\mathcal{A} \equiv^k \mathcal{B}$, $\mathcal{A} \models \phi$ if, and only if, $\mathcal{B} \models \phi$.

   (b) Show that the collection of finite linear orders is closed under $\equiv^2$, but not under $\equiv^2_p$ for any finite p. Explain why this shows that the preservation property from part (a) fails when restricted to finite structures.

2. We know that the substructure preservation theorem fails on finite structures. While it is unknown whether there is a syntactic characterisation of substructure closure on finite structures, we can obtain a semantic characterisation of the universal sentences.

   Say that a structure $\mathcal{A}$ is n-generated if there are $a_1, \ldots, a_n \in A$ such that $\mathcal{A}$ is the structure generated by $a_1, \ldots, a_n$. Show that the following are equivalent for any sentence ϕ:

   (a) ϕ is equivalent, on finite structures, to a $\forall$-sentence;

   (b) there is an n such that $\mathcal{A} \models \phi$ if, and only if, for every n-generated substructure $\mathcal{B}$ of $\mathcal{A}$, $\mathcal{B} \models \phi$.

3. This exercise is aimed at illustrating the use of infinitary methods to establish some of the inexpressibility results that were proved in the lectures using Ehrenfeucht-Fraïssé games.

   (a) Let I be the collection of axioms that states, for each n, that there are at least n distinct elements. Show that I is countably categorical. Deduce from this that there is no first-order sentence that is true in a finite structure if, and only if, it has an even number of elements.

   (b) Let QFLO be the theory which includes I along with the axioms of linear order with endpoints where each element other than the minimal one has a unique predecessor and each element other that the maximal one has a unique successor. Show that this theory has a countably saturated model. Deduce that there is no first-order sentence that is true in a finite linear order if, and only if, it has an even number of elements.

(c) Let Zs be the theory, in the language of graphs, with axioms stating that each element has exactly two neighbours and, for each n, there is no cycle of length n or less. Show that Zs has a countably saturated model. Deduce that there is no first-order sentence that is true in a finite graph if, and only if, it is connected.

4. *Craig's Interpolation Theorem* states that if ϕ is a sentence in the vocabulary σ, and ψ is a sentence in the vocabulary τ such that $\phi \to \psi$ is valid, then there is a sentence θ in the vocabulary $\sigma \cap \tau$ such that both $\phi \to \theta$ and $\theta \to \psi$ are valid.

   (a) Deduce from Craig's interpolation theorem that if ϕ is a sentence in the vocabulary $\sigma \cup \{<\}$ which is $<$-invariant on linear orderings of all σ-structures, then there is a σ-sentence θ such that $\models \phi \leftrightarrow \theta$.

   This fails on finite structures, as the next exercise show. Let σ be a vocabulary for Boolean algebras.

   (b) Show (using games) that there is no first-order sentence ϕ such that, for any finite Boolean algebra $\mathcal{A}$, $\mathcal{A} \models \phi$ if, and only if, $\mathcal{A}$ has an even number of atoms.

   (c) Show that there is a first-order sentence ψ in the vocabulary $\sigma \cup \{<\}$, where $<$ is new, such that, for any finite Boolean algebra $\mathcal{A}$ and any linear order $<$ on $\mathcal{A}$, $(\mathcal{A}, <) \models \psi$ if, and only if, $\mathcal{A}$ has an even number of atoms.

5. The aim of this exercise is to show that there is no first-order sentence ϕ in the language $\{<, E\}$ of ordered graphs which expresses, in an order-invariant way that a graph is connected.

   (a) Consider the graph $(\{0, \ldots, n-1\}, E)$ where $E(i, j)$ if, and only if, $i \equiv j + 2 \pmod{n}$ or $j \equiv i + 2 (\pmod{n})$. Show that this graph is connected if, and only if, n is odd.

   (b) Deduce that, if there were an order-invariant ϕ defining the connected graphs, there would be a sentence in the language of order whose finite models are exactly the orders of even length.