Stochastic Point Processes

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Distributions in Nature

[Filckr user Hans Dekker]

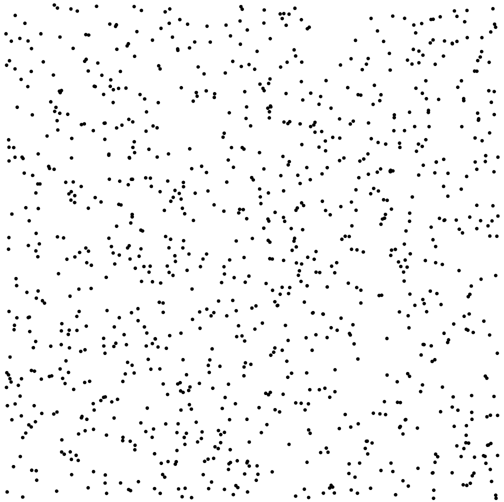
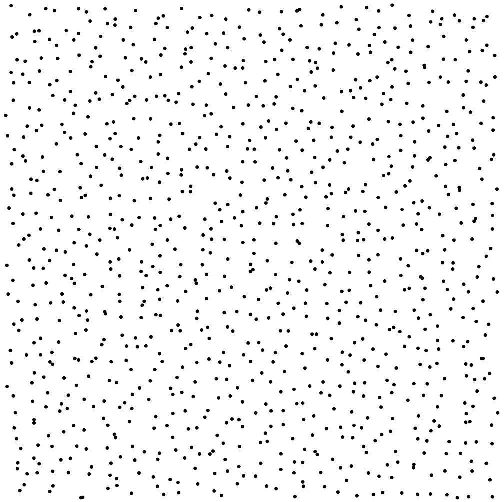
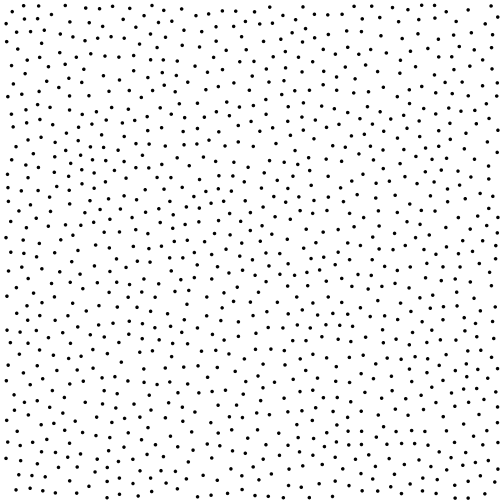
Sampling

- Conversion from continuous to discrete
- Integration in rendering

[Filckr user Josh Pesaver

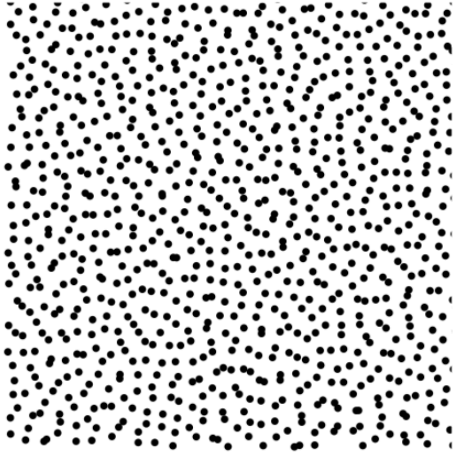
Point Distributions

- All kinds of patterns

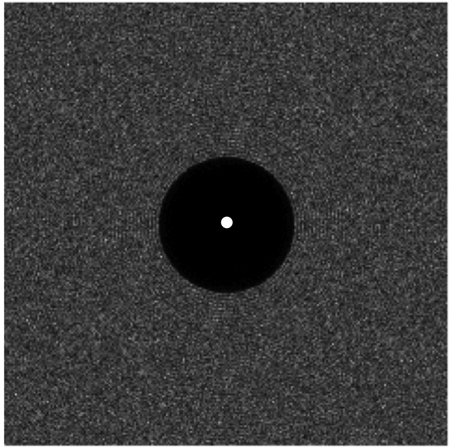
Studying Point Distributions

- How can we analyze patterns?

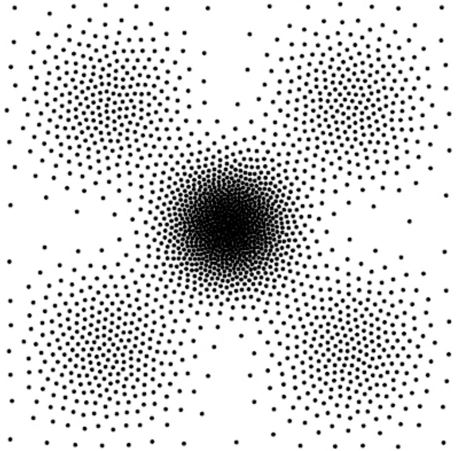


[Heck et al. 2013]

Periodogram



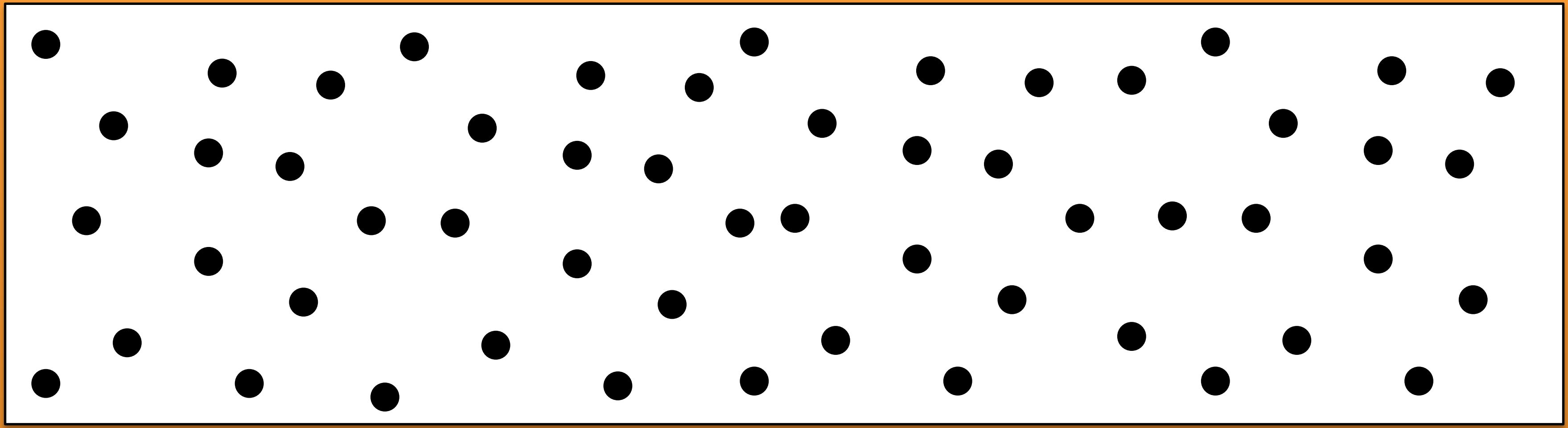
Density



[Fattal 2011]

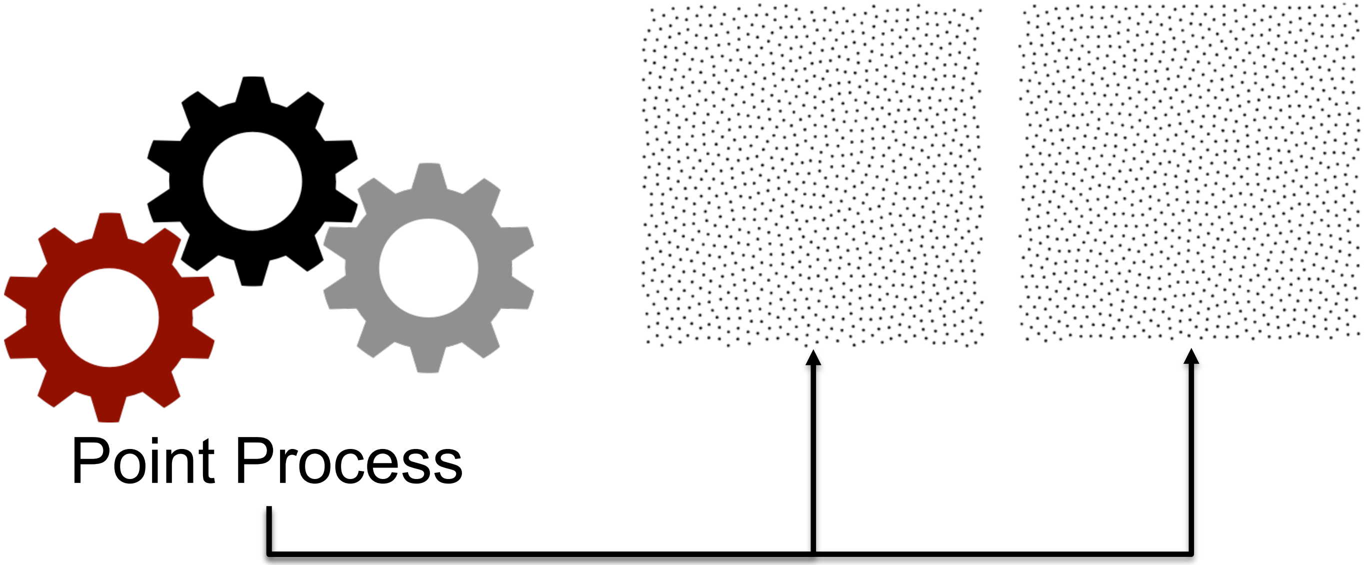
Point Processes

- Formal characterization of point patterns



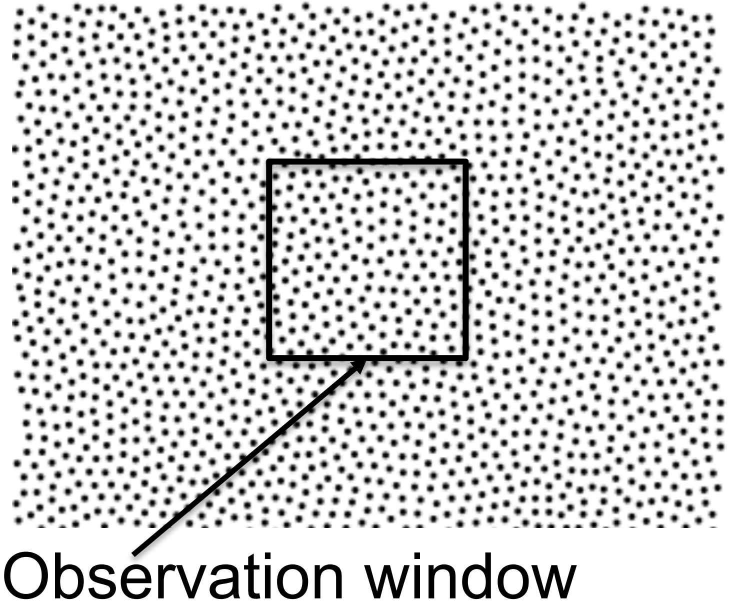
Point Processes

- Definition



Point Processes

- Infinite point processes

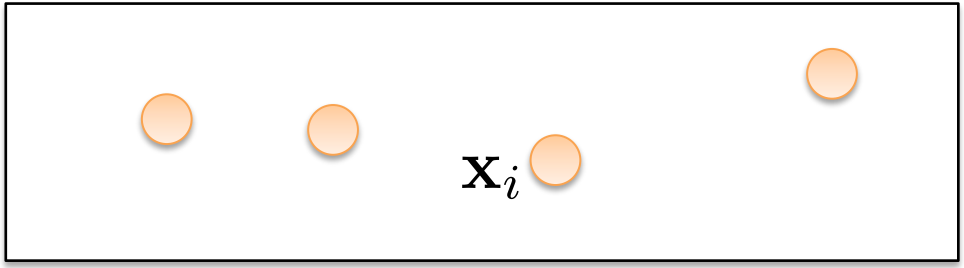


Point Process Statistics

- Correlations as probabilities

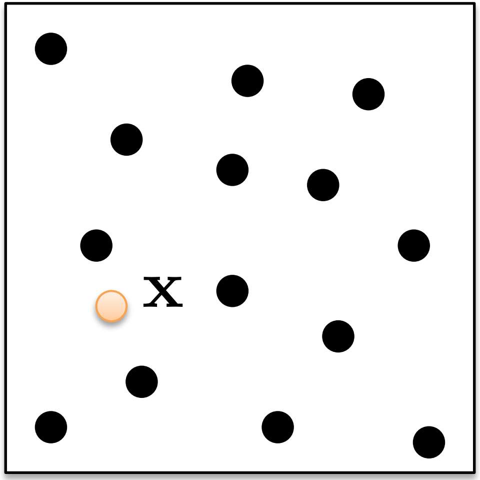
$$\varrho^{(n)}(\mathbf{x}_1, \cdots, \mathbf{x}_n)\, dV_1 \cdots dV_n = p(\mathbf{x}_1, \cdots, \mathbf{x}_n | N)$$

Product density — Small volumes — Points in space — Point process



Point Process Statistics

- Correlations as probabilities
  - First order product density



$$\varrho^{(1)}(\mathbf{x}) = \lambda(\mathbf{x})$$

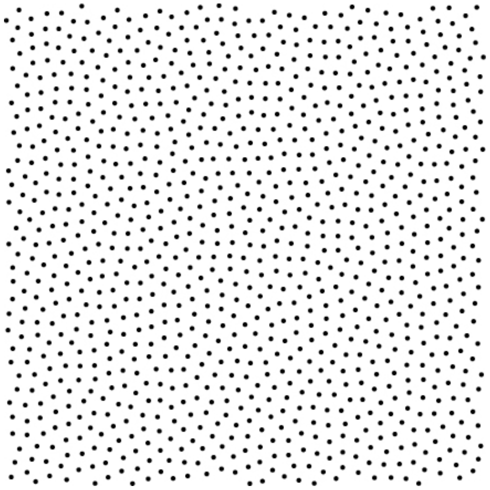
Expected number of points around x

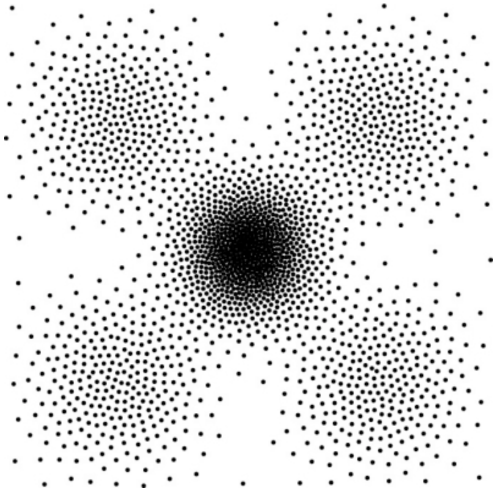
Measures local density

Point Process Statistics

- Correlations as probabilities
  - First order product density

$\lambda(\mathbf{x})$



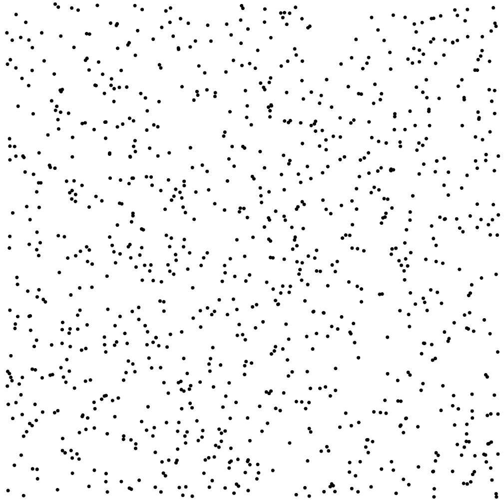


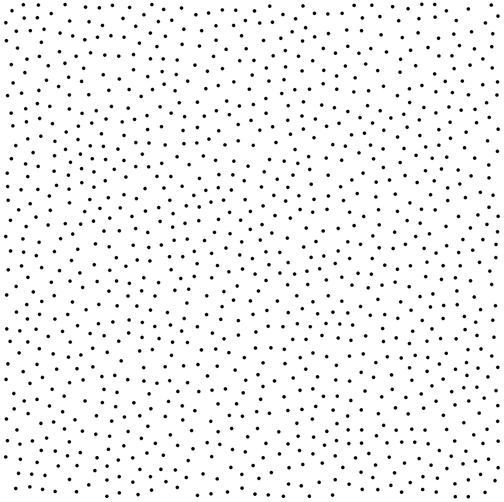
Point Process Statistics

- Correlations as probabilities
  - First order product density

$\lambda(\mathbf{x})$

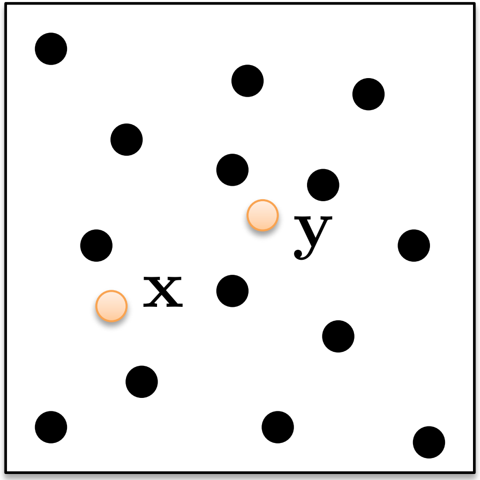
Constant





Point Process Statistics

- Correlations as probabilities
  - Second order product density



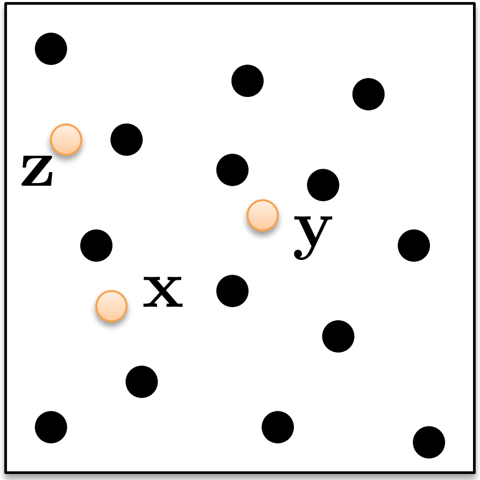
$$\varrho^{(2)}(\mathbf{x},\mathbf{y}) = \varrho(\mathbf{x},\mathbf{y})$$

Expected number of points around **x** & **y**

Measures the joint probability $p(\mathbf{x},\mathbf{y})$

Point Process Statistics

- Correlations as probabilities
  - Higher order?



$$\varrho^{(3)}(\mathbf{x}, \mathbf{y}, \mathbf{z})$$

Expected number of points around **x y z**

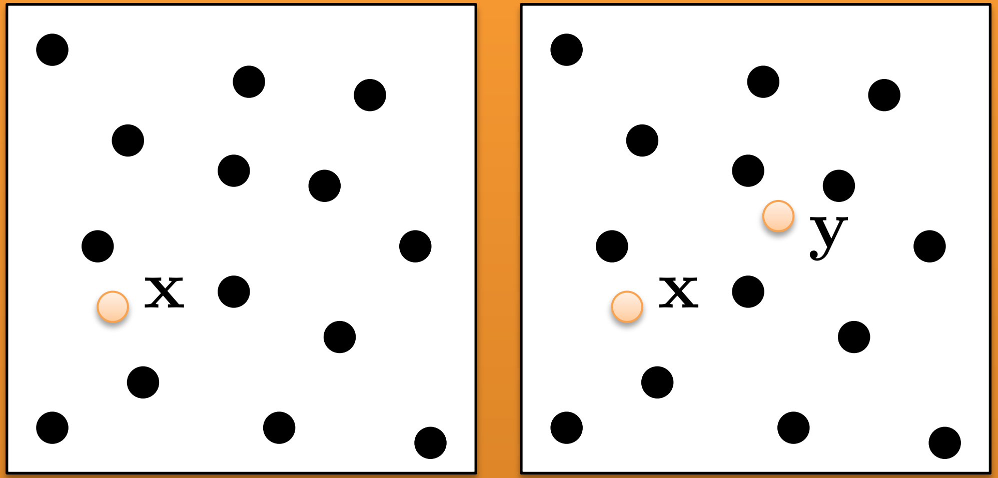
The “second order dogma” in physics

Point Process Statistics

- Summary:
  - First and second order product densities

$$\varrho^{(1)}(\mathbf{x}) = \lambda(\mathbf{x})$$

$$\varrho^{(2)}(\mathbf{x},\mathbf{y}) = \varrho(\mathbf{x},\mathbf{y})$$



Point Process Statistics

- Example: homogenous Poisson process
  - a.k.a random sampling

$$p(\mathbf{x}) = p$$

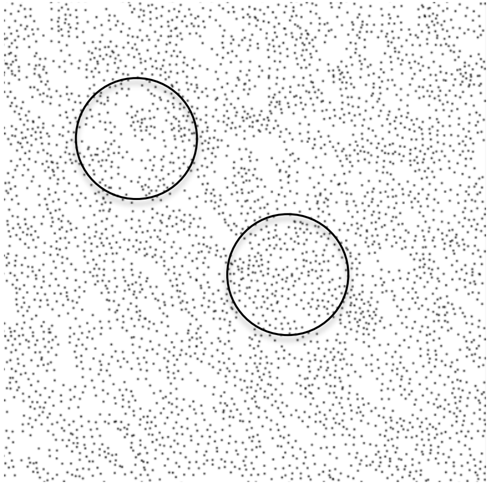
$$\lambda(\mathbf{x})dV = p$$
$$\lambda(\mathbf{x}) = \lambda$$

$$p(\mathbf{x}, \mathbf{y}) = p(\mathbf{x})p(\mathbf{y})$$

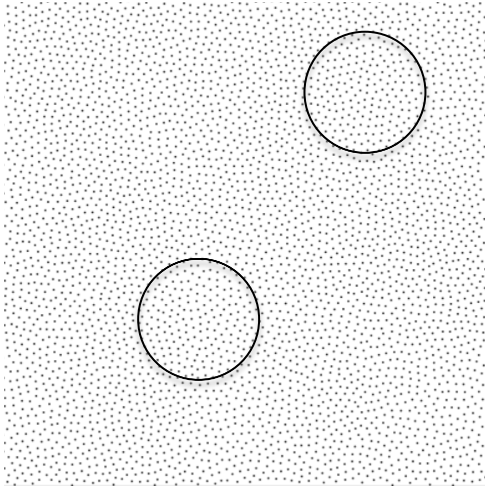
$$\begin{aligned} p(\mathbf{x}, \mathbf{y}) &= \varrho(\mathbf{x}, \mathbf{y})dV_x dV_y \\ &= p(\mathbf{x})p(\mathbf{y}) \\ &= \lambda(\mathbf{x})dV_x \lambda(\mathbf{y})dV_y \end{aligned}$$

$$\varrho(\mathbf{x}, \mathbf{y}) = \lambda(\mathbf{x})\lambda(\mathbf{y}) = \lambda^2$$

Stationary Processes



Stationary
(translation invariant)

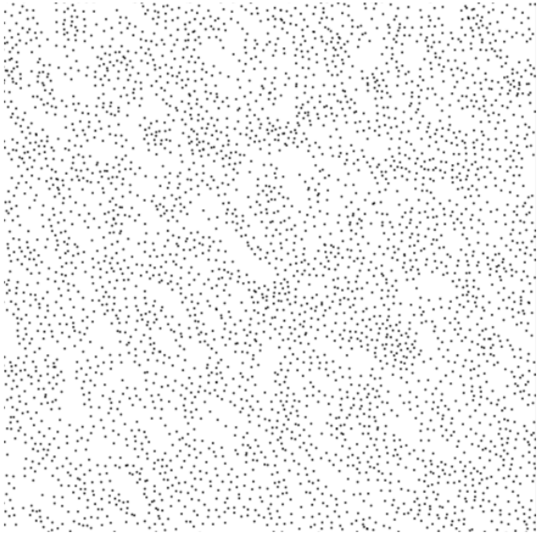


Isotropic
(translation & rotation invariant)

[Zhou et al. 2012]

Stationary Processes

- Stationary processes



$$\lambda(\mathbf{x}) = \lambda$$

$$\varrho(\mathbf{x}, \mathbf{y}) = \varrho(\mathbf{x} - \mathbf{y})$$

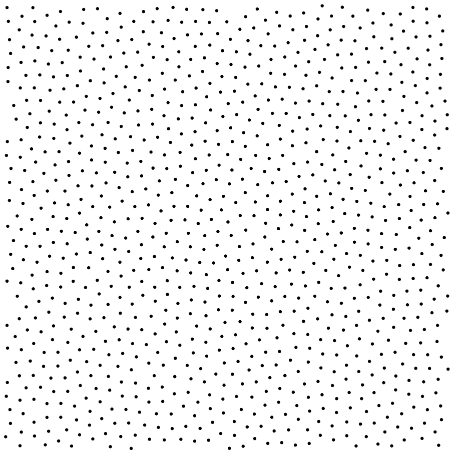
$$= \lambda^2 g(\mathbf{x} - \mathbf{y})$$

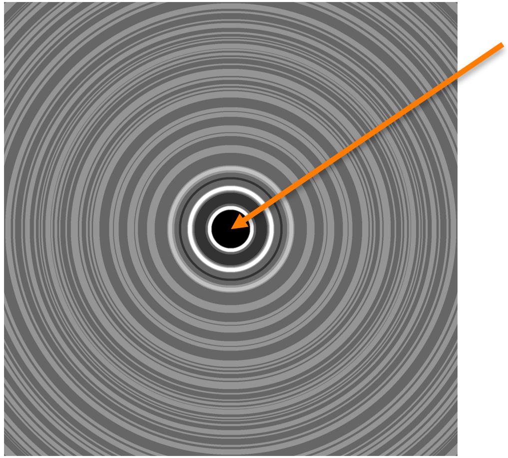
Pair Correlation Function (PCF)

DoF reduced from d^2 to d!

Stationary Processes

- Pair Correlation Functions $g(\mathbf{x} - \mathbf{y})$





PCF

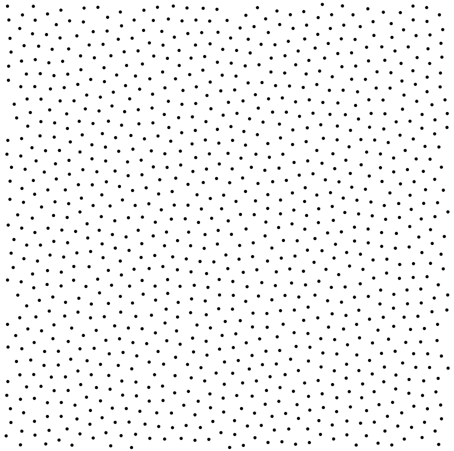
No points close to each other

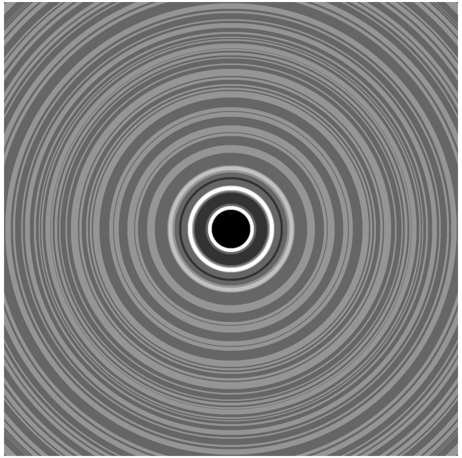
For isotropic processes: radially symmetric

Stationary Processes

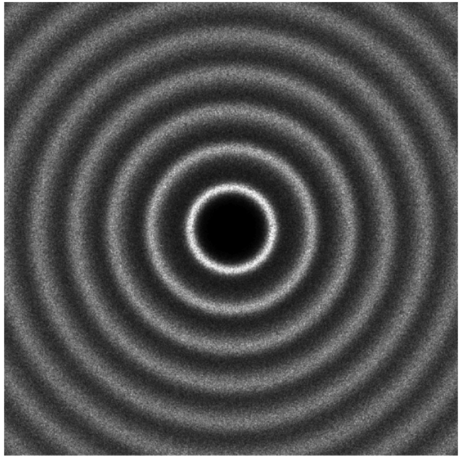
- PCF is related to the periodogram

$$\mathcal{F}\{g(\mathbf{h})\} = P(\omega)$$





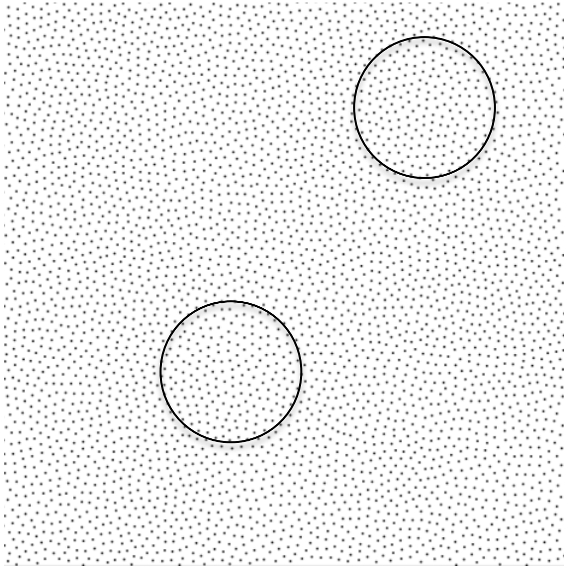
$g(\mathbf{h})$



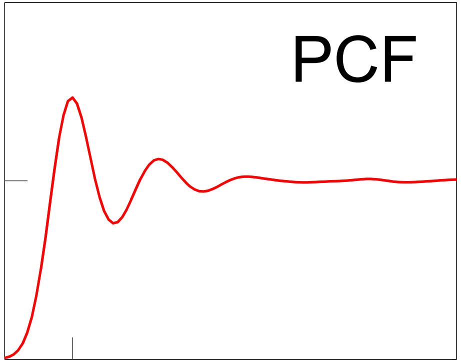
$P(\omega)$

Isotropic Processes

- Rotation invariant



$$\lambda(\mathbf{x}) = \lambda$$
$$g(\mathbf{x}-\mathbf{y}) = g(||\mathbf{x}-\mathbf{y}||)$$



Isotropic Processes

- Smooth estimator of the PCF

$$g(r) \approx \frac{1}{|\partial V_d| r^{d-1} \lambda^2} \sum_{i \neq j} k(r - d(\mathbf{x}_i, \mathbf{x}_j))$$

Volume of the unit hypercube in d dimensions

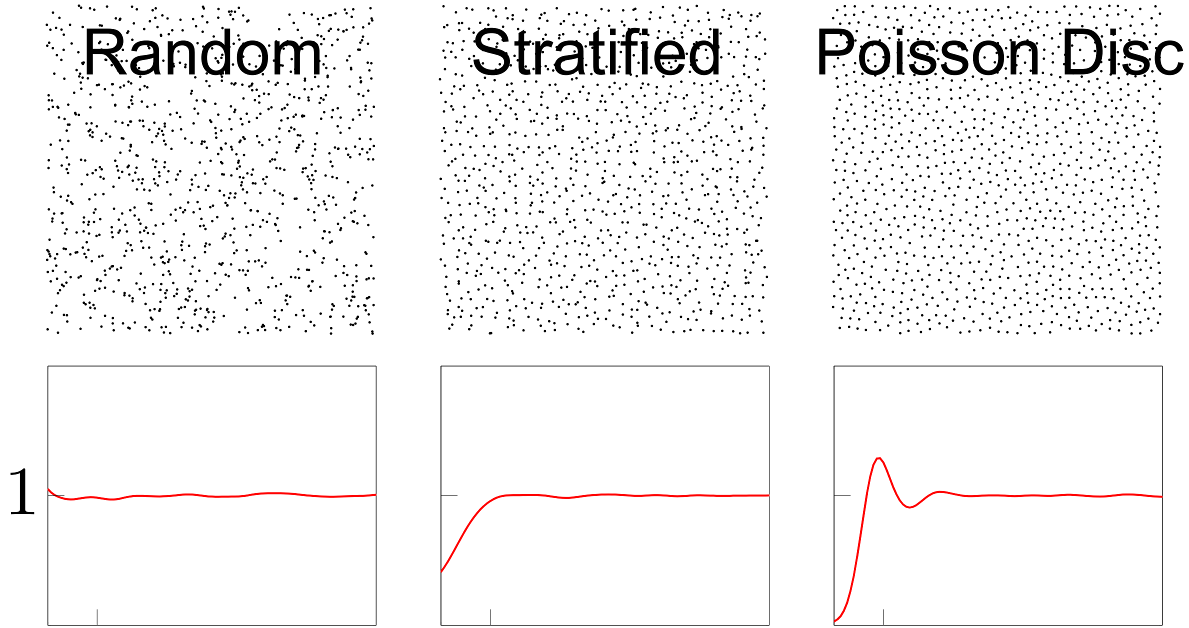
Intensity

Kernel, e.g. Gaussian

Distance measure

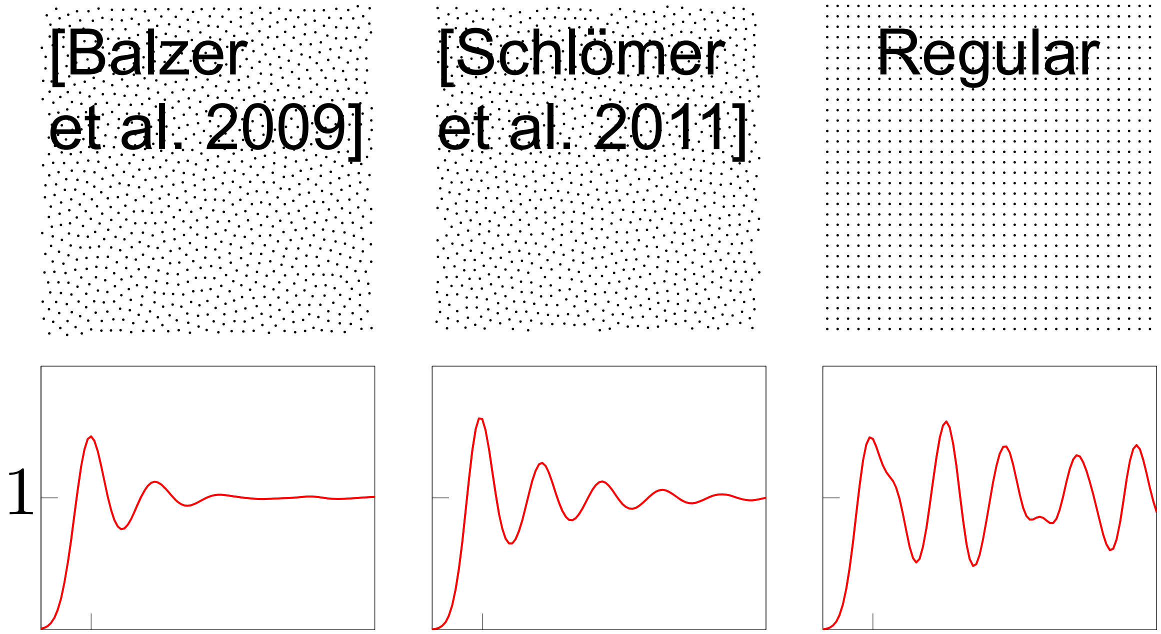
Isotropic Processes

- Estimated PCFs



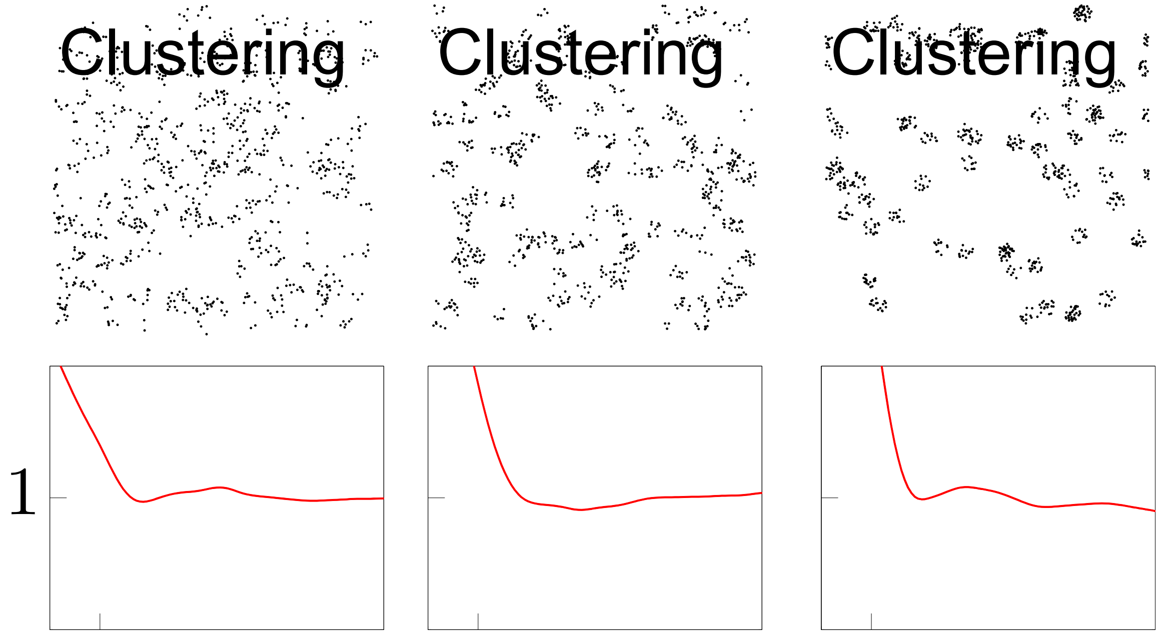
Isotropic Processes

- Estimated PCFs



Isotropic Processes

- Estimated PCFs



Reconstructing Point Patterns

[Oztireli and Gross 2012]

- Least squares fitting of point patterns

$$E(\mathbf{x}_1, \cdots, \mathbf{x}_n) = \int_0^\infty (g(r) - g_0(r))^2 \, dr$$

Sample Points

PCF of sample points

Target PCF

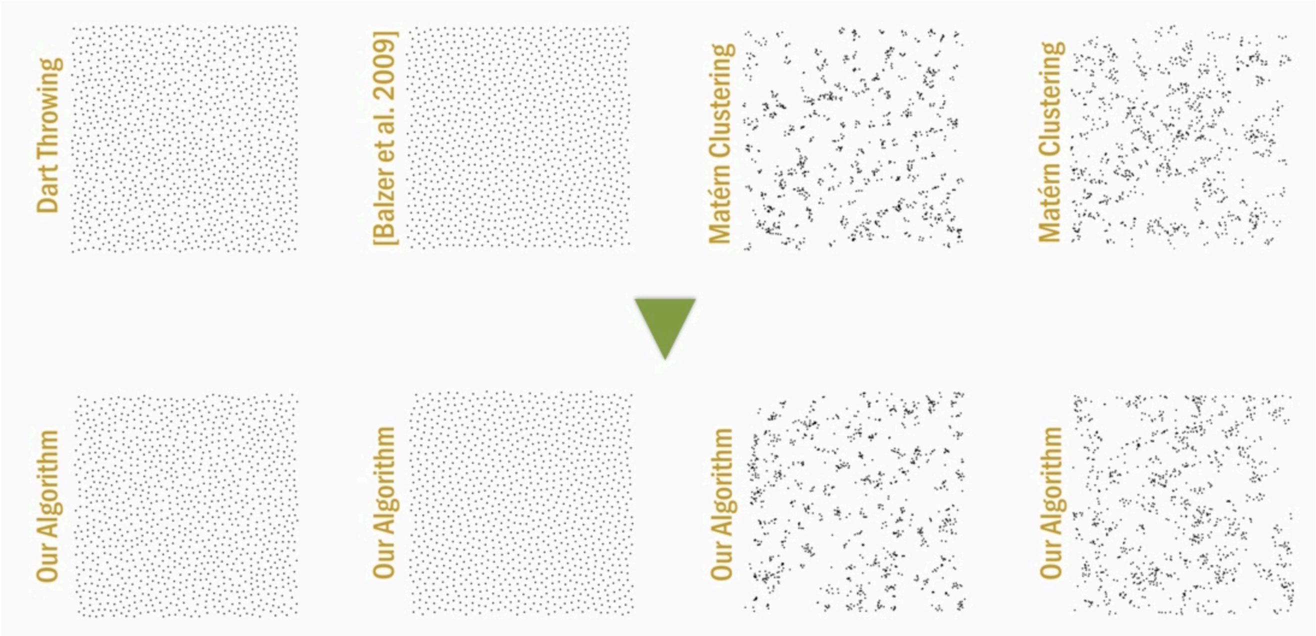
Gradient descend on the sample point locations

$$\mathbf{x}_i^{k+1} = \mathbf{x}_i^k - t \frac{\partial E}{\partial \mathbf{x}_i^k}$$

Reconstructing Point Patterns

[Oztireli and Gross 2012]

- Least squares fitting of point patterns

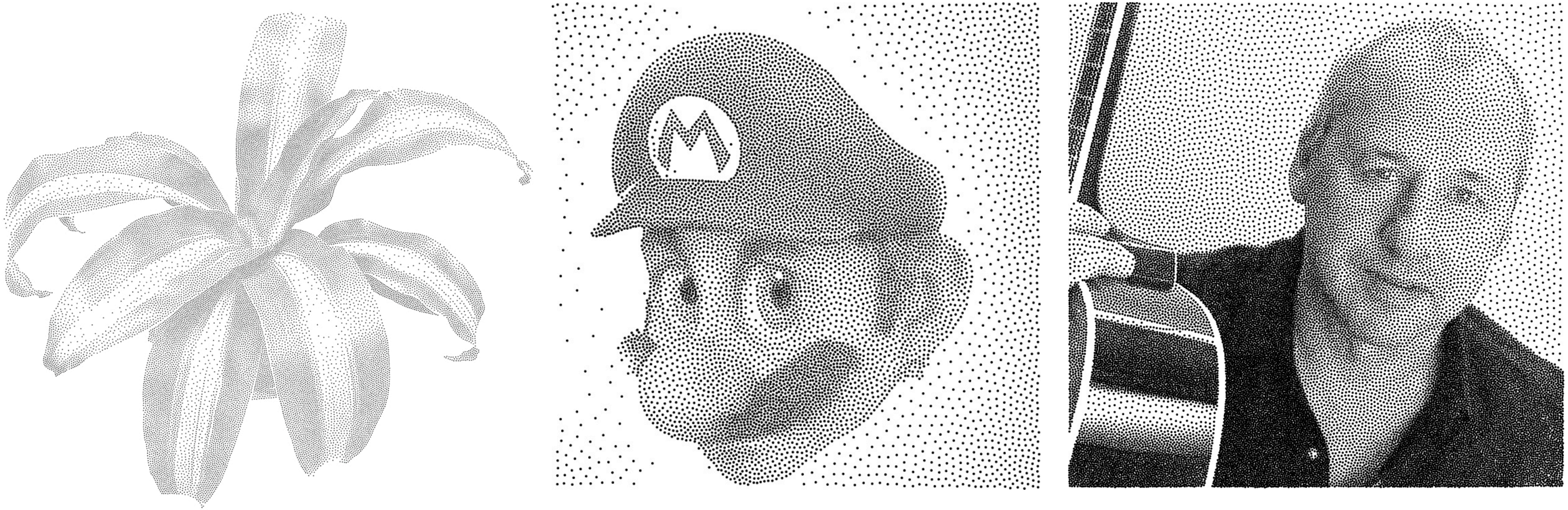


Reconstructing Point Patterns

- General isotropic patterns
  - [Wachtel et al. 2014]
  - [Heck et al. 2013]
  - [Zhou et al. 2012]
  - [Oztireli and Gross 2012]

General Point Processes

- What about: general point patterns

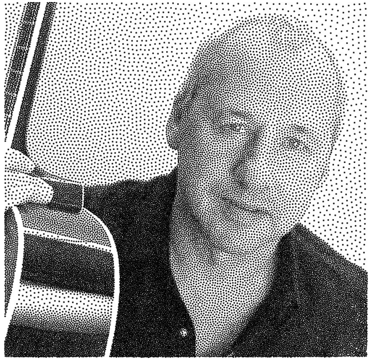


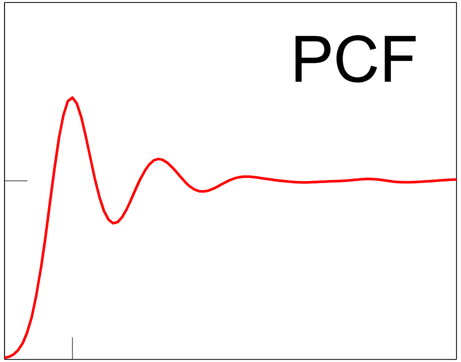
[Fattal 2011]

General Point Processes

- Idea 1: Use a tailored distance

$$g(r) \approx \frac{1}{|\partial V_d| r^{d-1} \lambda^2} \sum_{i \neq j} k(r - d(\mathbf{x}_i, \mathbf{x}_j))$$



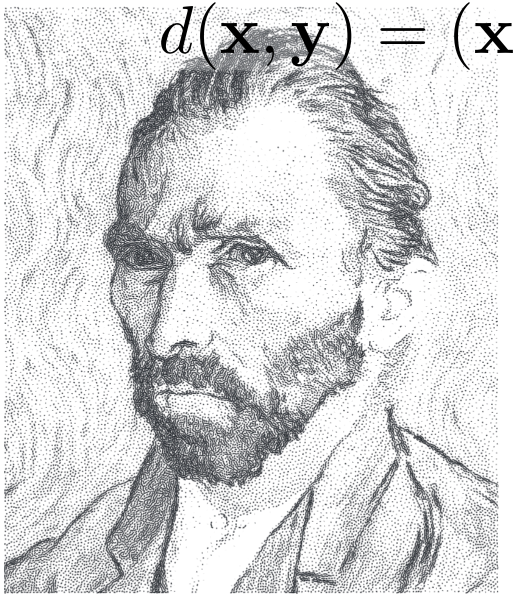


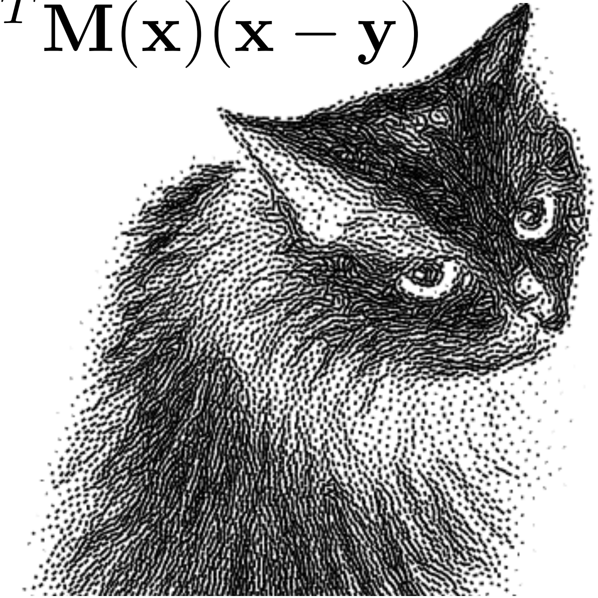
[Fattal 2011]

General Point Processes

- Idea 1: Use a tailored distance

$$d(\mathbf{x}, \mathbf{y}) = (\mathbf{x} - \mathbf{y})^T \mathbf{M}(\mathbf{x})(\mathbf{x} - \mathbf{y})$$

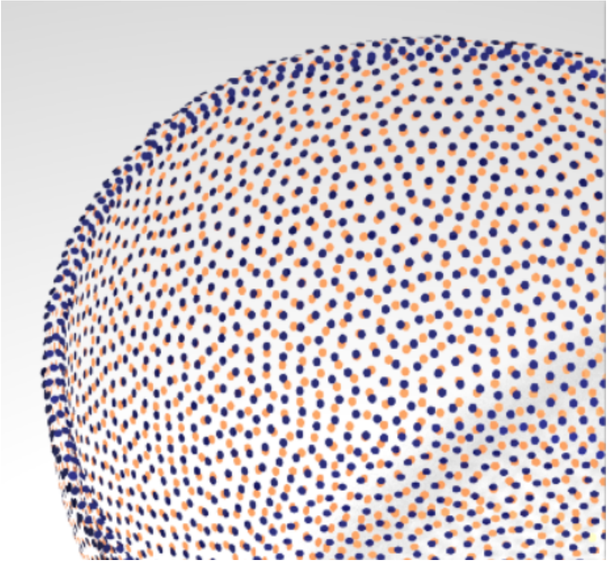




[Li et al. 2010]

General Point Processes

- Idea 1: Use a tailored distance

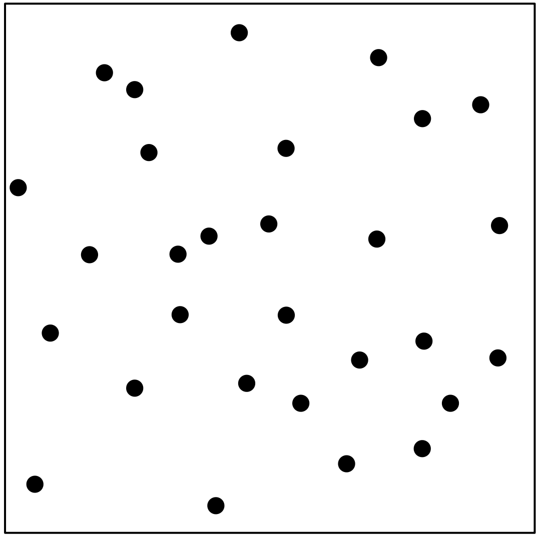


[Chen et al. 2013]

General Point Processes

- Idea 2: Change an initial point set

  Example: thinning according to a density

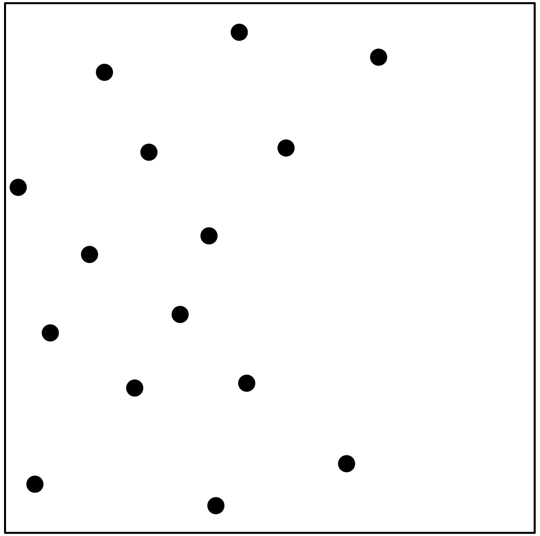


Start with an isotropic sampling
Remove each point with probability 1 - p(**x**)

General Point Processes

- Idea 2: Change an initial point set

  Example: thinning according to a density



Start with an isotropic sampling
Remove each point with probability 1 - p(**x**)

$$\lambda(\mathbf{x}) = p(\mathbf{x})\lambda$$
$$\varrho(\mathbf{x}, \mathbf{y}) = g(||\mathbf{x} - \mathbf{y}||)\lambda(\mathbf{x})\lambda(\mathbf{y})$$

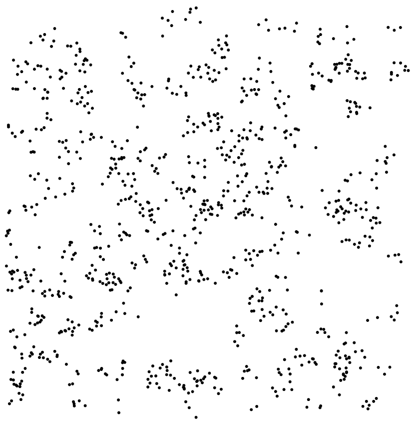
General Point Processes

- Idea 3: Define the probability of a sampling
  - Fall back to classical statistics
  - So far: infinite point processes
  - Now: finite with n number of points
  - Define the probability of a configuration:

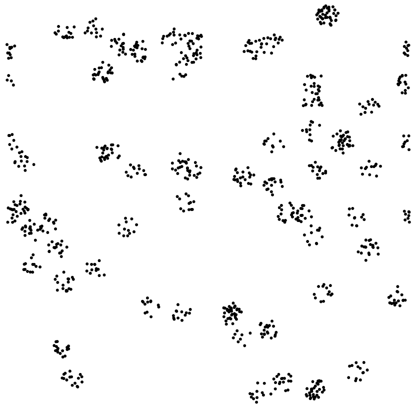
$$f(\mathbf{x}_1, \cdots, \mathbf{x}_n)$$

General Point Processes

- Idea 3: Define the probability of a sampling



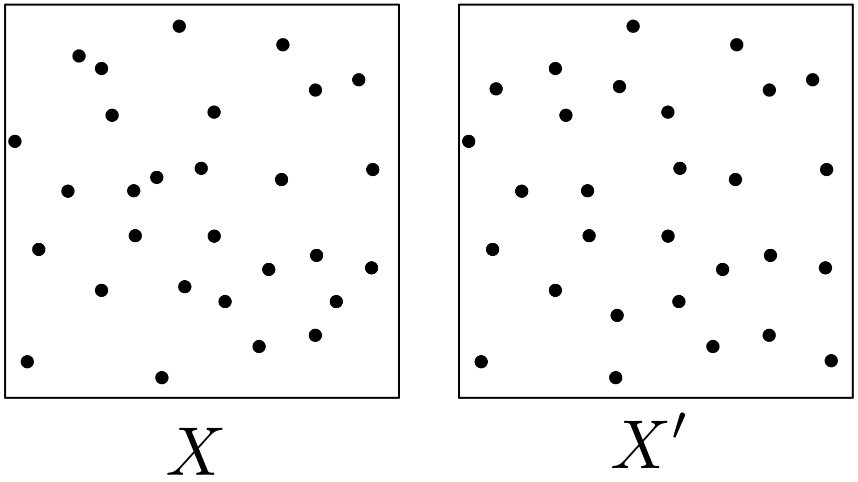
$$f(\mathbf{x}_1, \cdots, \mathbf{x}_n) = f_1$$



$$f(\mathbf{x}_1, \cdots, \mathbf{x}_n) = f_2$$

General Point Processes

- Idea 3: Define the probability of a sampling
  - Generation of point patterns - MCMC



Accept with probability

$$\min\left(1, \frac{P(X')M(X|X')}{P(X)M(X'|X)}\right)$$

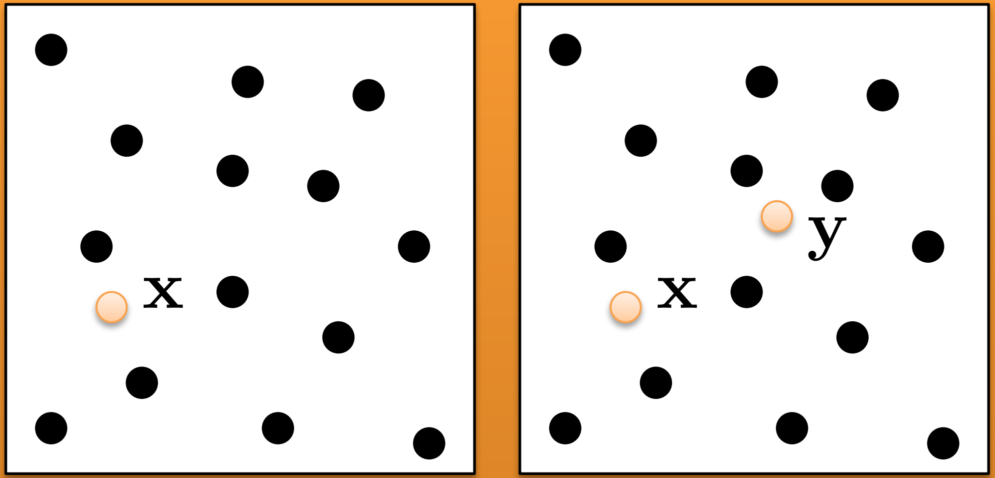
Proposal distribution

General Point Processes

- Idea 4: General correlations

$$\varrho^{(1)}(\mathbf{x}) = \lambda(\mathbf{x})$$

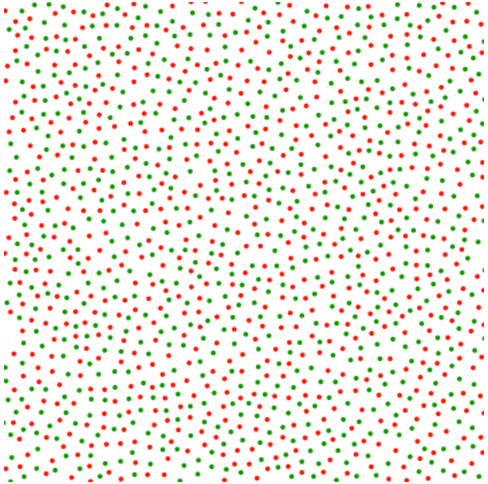
$$\varrho^{(2)}(\mathbf{x}, \mathbf{y}) = \varrho(\mathbf{x}, \mathbf{y})$$



Further Generalizations

- Marked point processes

Point locations + marks



PCF for class 1 $g_1(r)$

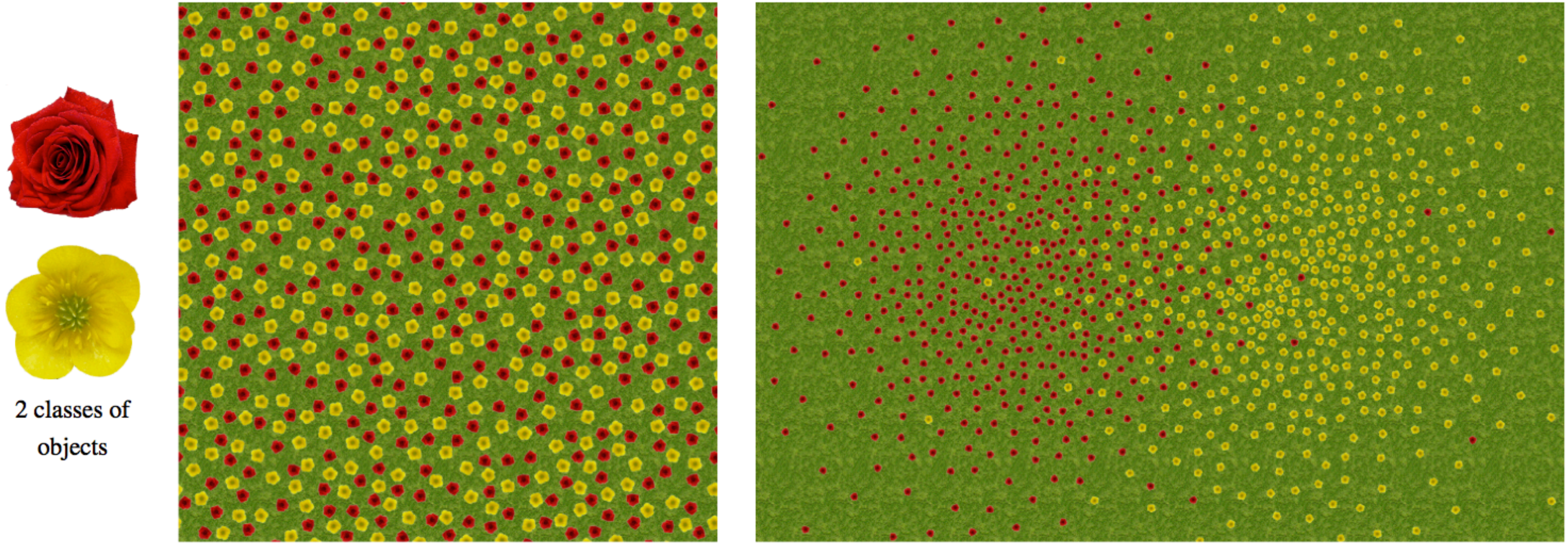
PCF for class 2 $g_2(r)$

Cross PCF for class 1 & 2 $g_{12}(r)$

$d(\mathbf{x}_i^1, \mathbf{x}_j^2)$ $\mathbf{x}_i^1 \in \mathcal{C}_1$ $\mathbf{x}_j^2 \in \mathcal{C}_2$

Further Generalizations

- Marked point processes
  - Discrete vs. continuous marks



[Wei 2010]

Further Generalizations

- Space-time point processes

[Ma et al. 2013]

Conclusions

- Stochastic point processes
  - Theoretical foundations
  - Explains general point patterns
  - Unified analysis and synthesis
  - Generalizes to measure spaces

Thanks

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