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First Year Report: *Game comonads, descriptive complexity & finite model theory*

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Introduction

As the title of this report suggests, my work this year is primarily concerned with three things: finite model theory, descriptive complexity and game comonads. Most of the pages which follow will detail what I've learned about these and how I believe I can develop them over the next two years of my PhD.

Finite model theory is the extensively studied mathematical theory of how finite objects relate to the logical sentences they satisfy. Descriptive complexity is one of the key twentieth century applications of this theory in computer science. It relates the expressive power of the logical languages needed to describe a class of finite objects with the computational power needed to recognise it. Game comonads are a brand new compositional perspective on descriptive complexity, recasting logical relations as approximations to homomorphism and identifying new connections throughout the field. In this report, I will demonstrate that this new perspective is rich and shows much promise but is as yet too limited to reimagine all of descriptive complexity theory and I will set out my plan for overcoming some of these limitations over the next few years and expanding the realm of compositional methods in descriptive complexity and finite model theory.

The report itself is broken down into four parts as follows:

I Literature Review

II Preliminary Research

III Summary of Proposed Dissertation

IV Timetable & Milestones

Part I, the largest part, will expand greatly on the brief sketch provided above, outlining the development of these three ideas throughout their existence, arriving at the current state of affairs and the opportunities for further research.
Part II will summarise the work I have done throughout the year both in learning about this area and exploring the boundaries of what is currently possible in the framework of game comonads.
Parts III and IV will provide a suggestive plan of how I intend to continue my research over the next two years.

1 Literature Review

Finite model theory is a vast area of computer science which studies the descriptive power of logics over finite models. As we will see in this brief introduction, this field straddles many areas including database theory, algorithms for constraint satisfaction and graph isomorphism and complexity theory. We will see that the restriction to finite models is not simply a practical convenience but adds new richness to the field mathematicians call model theory. We will see how classical approaches to this field have relied on a plethora of combinatorial tricks and tools. We will then see how the field has changed over the past twenty years with the emergence of two streams of developments which have at times extended the more combinatorial tools of the past. These streams, broadly speaking, are what I will call the algebraic and categorical (or (co)monadic) approaches to finite model theory. I will trace how each of these approaches has opened up new and exciting connections in FMT and between different areas of complexity and algorithms research but I will conclude that there is still work to be done in uniting these approaches. This will lay the foundation for the research I want to do in finding an expression of algebraic FMT methods in the comonadic framework of Abramsky, Dawar and Wang.

1.1 Finite model theory: its origins and why we study it

In the early twentieth century, before the establishment of anything resembling modern computer science, mathematicians were asking themselves questions the relationship between logical formulae (usually in first-order logic) and the objects or *models* which satisfy them. As anyone who has taken an undergraduate course in logic knows, the resulting meta-theory, model theory, is elegant and somewhat unexpected. We have, for one, a deep relationship between the syntax of first-order sentences and their semantics, as summarised in Gödel's completeness theorem. Here I state a later form of it due to Leon Henkin:

Theorem 1 (Henkin, 1947 after Gödel, 1929)**.**
If a first-order theory T is syntactically consistent (i.e. you can't deduce a contradiction from it) then T has a model

This forms the basis of a great number of results relating the syntax of theories to the semantic properties of their models. To list a few:

- **Łoś-Tarski Theorem**: A first-order-definable property is preserved under taking submodels if and only if it is definable in first-order logic using only universal quantifiers.
- **Lyndon's Positivity Theorem**: A first-order-definable property is preserved under surjective homomorphisms if and only if it is definable in first-order logic without negations.
- **Löwenheim-Skolem Theorem**: If a (countable) first-order theory has an infinite model then it has a model of size κ for eveny infinite cardinal κ

These theorems are clearly aesthetically pleasing and they form the beginnings of a long and fascinating tradition of model theory which continues today and which is excellently documented in Baldwin's history of the topic [8].

At this point, an impatient computer scientist (perhaps the reader) might ask "Why all this fuss about infinite cardinals" When we ask which models satisfy a first-order formula (or perhaps *query*) we hardly care about countable models let alone uncountable ones. What we really want to know is which finite inputs (graphs, systems of equations, database entries) are going to satisfy our query. So why not just restrict this elegant theory of models to finite ones and use the relevant results? Call this subfield the model theory of finite structures and be done with it. That would certainly make the writing of this review a lot simpler. However, it turns out that the the realm of finite models is quite different from its bigger sibling and this can for the most part be traced back to following result (due to Trakhtenbrot) and its main consequences (beginning with the failure of completeness). With this, we will soon see that the exploration of these differences warrants a whole new field of *finite model theory.*

Theorem 2. *Over a signature σ with at least one binary relation, the set of FO formulas ϕ which are finitely satisfiable is undecidable.*

Firstly why does this result show the failure of completeness? Well note that the set of finitely satisfiable sentences is clearly recursively enumerable (just go through each finite model in turn and halt if the sentence is satisfied) so we have decidability iff unsatisfiable sentences are recursively enumerable. So Trakhtenbrot shows that the set of sentences which are finitely unsatisfiable are not recursively enumerable. This precludes a finite version of the completeness theorem, as such a theorem would allow you to enumerate unsatisfiable sentences by enumerating proofs of false.

The proof of Trakhtenbrot's Theorem, which is given a nice exposition in Chapter 9 of Leonid Libkin's textbook on finite model theory [31], makes use of an encoding of an arbitrary Turing Machine M into a first-order sentence ϕ_M for which any model is a proof that M halts on the empty input. Then we derive undecidability from that of the Halting Problem. This type of close relationship between satisfiability over finite objects and computations halting in finite time is precisely the content of the next section which will introduce descriptive complexity theory.

1.2 Descriptive complexity theory

Descriptive complexity theory builds on the connection between finite model theory and computation established by Trakhtenbrot's theorem and has developed into a rich field studying the relationship between the logics used to state queries and the complexity of the algorithms available to solve them. This section will trace the development of this field. In particular, I will focus on the work done towards the central problem of finding a logic which *captures* (in a sense which I will define later) the complexity class P and the importance of finite variable and infinitary logics in this work. Throughout this section it should be clear that the language and techniques used in the development of descriptive complexity have quite a combinatorial feel. As you will see in following sections, it will be central to my thesis that we need a cleaner, more systematic, compositional way to capture these developments.

Fagin's Theorem and P vs. NP The question of whether logics and complexity classes can be related (which was raised by Trakhtenbrot's Theorem) was first answered in the affirmative in Ronald Fagin's 1973 PhD thesis [20]. Fagin's Theorem, which says that existential second order logic *captures*[1] NP, can be seen inaugural result in descriptive complexity theory. This result raises natural questions for complexity theorists, the most significant of which is whether there is a similar logical characterisation of P. If there is such a logic then comparing it with $\exists$SO will teach us about the relation between P and NP. If there is no such logic, we will know that $\mathsf{P} \neq \mathsf{NP}$. So, this paradigm of descriptive complexity and this question in particular of finding a logic to capture P are clearly a worthy areas of research. I will now sketch the progress made towards this goal since Fagin's Theorem launched this field in 1973.

FO isn't enough to capture P The natural place to start in this search for a logic capturing P is, as it were, at the bottom i.e. with first-order logic. This is a woefully inadequate language however which can't even capture simple polynomial properties such as counting and graph connectivity. This inadequacy is laid out plainly in the first few chapters of Libkin's textbook [31]. The main techniques for proving this weakness of FO over finite models are the game theoretic arguments introduced in the 1960's by Ehrenfeucht [19]. These *Ehrenfeucht-Fraïssé games* and the accompanying EF Theorem form a template for almost all the combinatorial techniques used to study the expressiveness of logics over finite relational structures.
The game itself works as follows. A spoiler and duplicator take turns marking elements in two relational structures A and B. Spoiler plays first on round i by picking a structure, say A, and marking an element $a_i \in A$. Spoiler responds by marking an element in the other structure, e.g. $b_i \in B$. By the end of the turn spoiler wins unless the marked pebbles form a partial isomorphism $a_i \mapsto b_i$ from A to B. The central theorem of [19] says that if spoiler has a winning strategy for the k round version of this game ($A \equiv^k_{EF} B$) then A and B agree on FO sentences of *quantifier rank* at most k ($A \equiv_{L_k} B$). This gives us a way of showing that some property $\mathcal{P}$ of finite relational structures is not expressible in FO, namely we find for each k there exists $A_k \in \mathcal{P}$, and $B_k \notin \mathcal{P}$ s.t. $A \equiv^k_{EF} B$. In the case of the EF games, we can show that, for example connectivity is not an FO-definable property of graphs. This technique of using combinatorial games to determine the expressiveness of logics has been a central theme of descriptive

[1] In the sense that for any set S of models whose decision problem is in NP there is a $\phi_S \in \exists$SO s.t $S = \{\mathbb{M} \mid \mathbb{M} \models \phi\}$ and for any $\phi \in \exists$SO the decision problem for $S_\phi = \{\mathbb{M} \mid \mathbb{M} \models \phi\}$ is in NP.

complexity theory and creating a unified perspective on these methods is one of the aims of my work.

Recursion, fixed-points and infinitary FO Motivated by the example of connectivity in graphs, a natural extension of FO is to try and introduce some form of recursive definition. The way this has been handled in logic is with the addition of fixed-point quantifiers to first-order logic as first explored independently by Immerman [27] and Vardi [34]. These operators allow for the recursive definition of new predicates such as $\mathbf{ifp}(\phi(R, \bar{x}, \bar{y}))(\bar{x})$. We call the extension of FO by these operators, IFP or inflationary fixpoint logic. This logic can easily capture connectivity and reachability queries that were impossible in FO logic but the new syntax makes it difficult to use the same game techniques introduced in the last section. The solution is to "unravel" IFP and contain this finitary fixed-point logic inside an infinitary version of FO logic. The first naive attempt to do this is $L_{\infty\omega}$, the logic of FO sentences closed under infinitary conjunctions and disjunctions. This, however, is far too strong. Note, in particular, that any relational structure A has a $\phi_A \in$ FO s.t. $B \models \phi_A \implies A \cong B$. So by taking the disjunction of all the appropriate ϕ_A, $L_{\infty\omega}$ can express all properties of finite relational structures, *even* undecidable ones! The solution, which we'll see in the next paragraph is the introduction of finite variable logics, a key development in descriptive complexity and an important extension of the realm of applicability of using games to reason about logics.

Finite variable infinitary logics give us some hope As pointed out at the end of the last paragraph, IFP is contained in $L_{\infty\omega}$ but this containment can't tell us much as $L_{\infty\omega}$ is too powerful over finite models. Instead a restriction must be found for which we can develop a notion of the EF game. This restriction is to the finite variable form $L^{\omega}_{\infty\omega} = \bigcup L^k_{\infty\omega}$, where $L^k_{\infty\omega}$ is infinitary FO sentences on no more than k variables (where variables can be reused). The development of this theory in the FMT community is covered extensively in Kolaitis and Vardi's expository work [30]. A key part of this track of work was the characterisation of $L^k_{\infty\omega}$ by a variant of the EF game where instead of limiting the number of turns, we limit the marked positions to k movable "pebbles". This k-pebble game allows us to answer the question of whether IFP captures P. As we'll see this is not the case but the field of finite variable logic proves a fertile area of research and the extension of these pebble games with various additional operators has led to descriptions for increasingly powerful logics.

Still IFP and IFP+C are not enough The story of these developments unfolds throughout the 80's and 90's and is reviewed comprehensively by Grohe in his 1998 paper [23]. Early positive developments in this realm include Immerman and Vardi's independent proofs that IFP captures P over finite totally ordered structures and Kolaitis and Vardi's discovery of a 0-1 law for $L^{\omega}_{\infty\omega}$[2], and by comparison also for IFP. The finite variable approach can also be used to

[2] which states that for $\phi \in L^{\omega}_{\infty\omega}$, either $A \models \phi$ for almost all A or $A \not\models \phi$ for almost all A

prove negative results about extensions of FO. In particular, it is used to show that IFP cannot capture P over all finite structures (by showing that parity is not expressible in $L^{\omega}_{\infty\omega}$). This then suggests the next extension of FO which was posited to capture P, namely IFP + C, the extension of IFP with counting quantifiers $\exists^{\geq n}$. In a similar manner to the bounding of IFP by $L^{\omega}_{\infty\omega}$, this power of this new logic can be bounded above by infinitary finite variable logic with counting quantifiers $C^{\omega}_{\infty\omega}$. Analysis of this logic was then enabled by modified versions of the k pebble games introduced by Immerman and Lander [28] and Hella [25]. The hope that IFP+C would capture P was then dashed by the construction of Cai, Fürer and Immerman in [14]. This paper demonstrated a graph property which which is decidable in P but is not expressible in $C^{\omega}_{\infty\omega}$. The process for deciding this graph property is essentially a canonisation algorithm based on the efficient computation of linear algebraic rank and so it is no surprise that the challenge of going beyond IFP+C has been answered in large part over the last two decades by the addition of linear algebraic operators to IFP+C. This will be covered in the following paragraph.

Beyond IFP+C: Linear Algebraic Logics At this point in our story we catch our first glimpse of a phenomenon that my work will aim to better understand. That phenomenon is the application of algebraic techniques to finite model theory and descriptive complexity. This first becomes an important thread in the search for a logic capturing P when Dawar et al. introduce rank logic in [17]. This new logic, FPR, extends IFP with operators for computing the rank of $n^r \times n^r$ matrices over the field $\mathbb{F}_p$ for some p. By varying r and p this is shown to generalise IFP+C and to express the the previously inexpressible queries underlying the CFI construction. This makes it a good candidate for a logic capturing P. To analyse the limits of this new logic, the theory of pebble games is extended by Dawar and Holm to include algebraic rules that make it harder for spoiler to win, an overview of which is provided in [18]. These pebble games are used by Holm in his PhD thesis [26] to rule out FPR capturing P for $r = 1$ but left open the possibility that FPR could capture P for some parameter r and some choice of prime q. This possibility is ruled out by Grädel and Pakusa in [22] who show, using non-game theoretic techniques that for all arities FPR over q and FPR over p are incomparable and so none of these capture P. Furthermore, very recent work by Dawar, Grädel and Pakusa [16] shows that even with the addition of *all* linear algebraic operators (over finite fields $\mathbb{F}_q$ for $q \in Q$) as seen in $\mathbf{LA}^{\omega}(Q)$, we can't capture P if Q is not the set of all primes. Additionally, they show that this logic is the logic captured by Dawar and Holm's k-invertible-maps game[26] over the set of primes Q. This extension by linear algebraic operators remains an object of interest in descriptive complexity theory. For example, it is not yet known if the logic with operators over all prime fields capture P.

1.3 Categorical methods in finite model theory

Having traced the development of the fields of finite model theory and descriptive complexity theory in the last two sections, I will now introduce the categorical methods which are central to my work in these areas. I'll outline how the introduction of the pebbling comonad by Abramsky, Dawar and Wang in [2] and the extension of this framework by Abramsky and Shah [4] have given us a rich compositional perspective on the fragments of FO which are important for finite model theory, exposing new connections between purely combinatorial and logical approaches of the past. I'll show that this framework, while powerful, is not yet equipped to describe some the more recent algebraic developments in descriptive complexity (some of which we saw in the previous section) and other algebraic techniques in related areas. This, I hope will set the scene for the work I have been doing this year and for my planned research over the next two years.

Homomorphisms in finite model theory The study of homomorphisms between finite relational structures has been of interest to researchers in complexity theory, implicitly or explicitly, for quite a long time. Indeed, two central problems studied in complexity theory, namely, the graph isomorphism (GI) and constraint satisfaction problems (CSPs) can be stated in terms of homomorphisms. GI asks for any two (finite) graphs if there exists an invertible homomorphism (isomorphism) between them, while CSP can be phrased as the following decision problem: for a signature σ and domain $\mathbb{A}$ over σ, and some $\mathbb{B}$, a relational structure over σ (seen as a set of variables with some constraints) decide whether there is a homomorphism $\mathbb{B} \to \mathbb{A}$ (seen as a satisfying assignment of the variables in $\mathbb{B}$ to elements in $\mathbb{A}$.) It seems natural then to consider as central to finite model theory the category $\mathcal{R}(\sigma)$, with finite relational structures over signature σ as objects and homomorphisms between them as maps. However, one issue with this is evident throughout the last section. Much of the work done in investigating the power of logics over finite structures is done not through studying homomorphism or isomorphism between structures but rather the weakened versions of these relations engendered by the various combinatorial games described above. These games give us useful (and crucially polynomial time computable) approximations to isomorphism and homomorphism in $\mathcal{R}(\sigma)$. This in turn leads to a way of studying algorithms for the CSP and GI problems. The challenge of relating these approximations to the underlying category $\mathcal{R}(\sigma)$ and developing a systematic theory of these approximations has been, until recently, unexplored.

The Pebbling Comonad (& Friends) The first part of this challenge is answered by Abramsky, Dawar and Wang's seminal paper [2] on the pebbling comonad. This paper introduces a comonad, $\mathbb{T}_k$ which in a sense captures the k pebble game. $\mathbb{T}_k$ sends a relational structure to the set of histories of pebble positions over that structure with relations induced on those tuples of histories which are consistently ordered and whose pebbled positions are related in the original structure. This construction on $\mathcal{R}(\sigma)$ "captures the k pebble game, in the sense that the morphisms in the coKleisli category category of $\mathbb{T}_k$ correspond to winning strategies for duplicator in the one-way k pebble game, i.e. a map $\mathbb{T}_k A \to B$ tells duplicator where to play their pebble in B in response to every history of moves of spoiler on A. Abramsky, Dawar, and Wang then show how standard constructions in this category theoretic framework have interesting (and somewhat surprising) finite model theory interpretations. For example, the isomorphisms in this coKleisli category turn out to be strategies for the k-bijection game and coalgebras of $\mathbb{T}_k A$ are tree decompositions of A witnessing treewidth$(A) < k$. These new formal connections between classical finite model theory techniques and parameters gives a sense of the richness of this comonadic perspective and we will see later how it is being used presently to reinterpret finite model theory results and algorithms for CSP and GI.

Faced with the second part of this "challenge", to develop a systematic framework of tractable approximations to homomorphism and isomorphism, the main source of progress we have is work done by Abramsky and Shah in [4]. This uses the pebbling comonad as a template for constructing and analysing a number of different game-theoretic approximations to homomorphism found in finite model theory. The other two games in this article, alongside the k pebble game, are the k round Ehrenfeucht-Fraïssé and modal bisimulation games. In each case, they give an account of how the comonad can be used to encode very similar connections as were seen in the pebbling comonad. This establishes a systematic way of analysing some of the basic logical games at the heart of finite model theory and descriptive complexity.

Limitations of Game Comonads The framework provided by game comonads is satisfying and elegant but, as of yet, relatively limited. Indeed, as outlined by Abramsky and Shah, it thus far only accounts for games used to test logics weaker than IFP + C, which we know is not enough to analyse all polynomial time approximations to homomorphism. In particular, this framework stops short of describing games with linear algebraic rules, such as Holm and Dawar's IM-equivalence game and thus cannot be used to analyse logics which may capture P such as LA(Q). A major open problem in this field then is how to extend this category-theoretic framework to capture such more powerful logics. This is what I hope to answer in the course of my research and in the following section I'll show why answering this question is likely to provide insight into some hot topics in logic, algorithms and complexity theory.

1.4 Related fields and the outlook for category-theoretic finite model theory

So far in this review, I have outlined the finite model theory underpinning my proposed work, traced the development of the central question in descriptive complexity theory of whether there are logics capturing P over finite and introduced the new and somewhat incomplete category-theoretic approach to this question. In this section, I will review some recent advances in related areas of finite model theory, algorithms and complexity theory and some other categorical perspectives emerging in the field. My hope is that my work in expanding the language of game comonads will develop new connections and a deeper understanding of some of these areas. The developments which I think are most relevant, and which I'll focus on here are recent breakthroughs in algorithms for CSP and GI, new homomorphism preservation theorems in finite model theory and novel uses of monads in other areas of complexity and model theory. As my hope is to extend the comonadic framework to capture games with algebraic rules, I will lay an emphasis in this section on the wealth of results in these related areas which have a similar algebraic flavour.

New algorithms for CSP and GI CSP and GI are central problems studied in algorithms and complexity theory and both have experienced recent high-profile breakthroughs: in GI a quasi-polynomial time algorithm was found by Babai [7] and in CSP two independent (though as yet unverified) proofs, by Bulatov [12] and Zhuk [35] have emerged for the long-standing Dichotomy Conjecture, stating that all instances of CSP are in P or NP-complete and introduced in terms of CSPs by Feder and Vardi in [21]. As mentioned in the previous section, both GI and CSP can be expressed as questions about maps in the category $\mathcal{R}(\sigma)$. Indeed, as noted by Abramsky et al. [2], the $\mathbb{T}_k$ comonad provides an interesting connection between k-local consistency methods for CSP and k-Weisfeiler-Lehman methods for GI, both of which are approximate solutions to the respective problems which run in polynomial time and correspond to exact solutions over classes of structures with bounded treewidth. In particular, the coKleisli morphisms $\mathbb{T}_k A \to B$ represent k-locally consistent solutions to the instance A of CSP(B) while the coKleisli isomorphisms $A \cong_{\mathcal{K}} B$ correspond to the structures A and B being deemed isomorphic under the k-Weisfeiler-Lehman process.

This connection is interesting but these two approximations are far from state-of-the-art in CSP or GI. Recent developments in CSP research, culminating in the resolution of the Dichotomy Conjecture, make clear that there is much more to be understood in this framework and give some indications of how this might be done. This program of work, centres around the idea, introduced by Jeavons, Cohen and Gyssens in [29] and developed by Bulatov, Jeavons and Krokhin in [11], that domains $\mathbb{B}$[3] which admit non-trivial higher-order symmetries (polymorphisms) are precisely the ones for which CSP($\mathbb{B}$) are tractable. This line of work has effectively classified instances of CSP, up to L-reduction, based on their algebras of polymorphisms. Powerful techniques such as Barto and Kozik's Absorption Lemma [9] have reduced the tractable classes to essentially two. The first is the problems for with *bounded treewidth* (solvable by k-local consistency methods) and the second is those with *few subpowers*. This

[3]Explain the caveats that go with this

second grouping, which are tractably solvable by an algorithm which essentially generalises Gaussian elimination, is one which currently doesn't have a neat exposition in the comonadic framework of Abramsky, Dawar et al. I believe that finding such an exposition would help to better understand the work done in resolving the Feder-Vardi Conjecture and perhaps relate this to tractable cases for GI. A good starting point for this line of work is the work of Atserias, Bulatov and Dawar from 2009 [6] which shows that the domains of these few subpowers cases of CSP cannot be captured in IFP+C.

Homomorphism preservation theorems Another and perhaps more obvious possible advantage of viewing finite model theory through the compositional framework of comonads, is gaining a more natural understanding of the interaction between homomorphisms and first-order sentences over finite models. Interest in this area of finite model theory has been reignited by Rossman's spectacular proof in 2005 [33] that the homomorphism preservation theorem is one of the few results of classical model theory which remains true over finite models. Rossman's proof of this result involves some inspired combinatorial constructions and is still feels a bit mysterious (at least to me). It is hoped that game comonads may provide a more systematic way of explaining this and other similar results. This is the motivation behind Abramsky and Paine's recent note on *Functorializing Rossman* [3], which starts this process of formalising Rossman's proof in this framework. Other finite homomorphism theorems such as Otto's Finite Guarded Invariance Theorem proved in [32] may also admit such interpretations in our framework. In fact, curiously enough, this proof has an algebraic flavour, relying at a crucial point on some group theoretic constructions. This seems to suggest that to truly understand preservation theorems in the comonadic framework we may first need to answer our outstanding questions about enhancing it to encompass the more powerful algebraic logics discussed above. I will discuss these ideas further later when I discuss my recent project on homomorphism preservation theorems in Section 2.1.

Other category-theoretic horizons in finite model theory I'd like to conclude this section and this literature review by touching on two other very recent applications of category theory to the study of finite relational structures. These are Abramsky, Barbosa, de Silva and Zapata's *Quantum Monad* [1] and Bojańczyk's *Two Monads for Graphs* [10].

The first of these gives a categorical representation of the notion of a *quantum homomorphism* between relational structures, a game approximating homomorphism wherein two separated players (with access to some d shared qubits) aim to convince a verifier of the existence of a homomorphism between the structures. This is done via a monad $\mathbf{Q}_d$ for which Kleisli morphism coincide with d-quantum homomorphisms in much the same way as $\mathbb{T}_k$ captures k local homomorphisms described previously. It seems very natural then that these constructions should be seen as duals and indeed Abramsky and Dawar, in a recent grant proposal suggest using the terminology of *resources* for such monads and *coresources* for such comonads. It thus could be important to understand any developments made in this sister field or even to relate these ideas formally (as I have tried to do this year in work I will describe in Section 2.3).

The second application, that of Bojańczyk, is less obviously related to the game comonad but I see it as providing a interesting alternative to the set-up of monads/comonads on $\mathcal{R}(\sigma)$, while also providing an elegant description of a famous result in finite model theory. This paper [10] provides a novel category theoretic exposition of Courcelle's Theorem[4]. In the course of doing this, Bojańczyk uses monads to construct hypergraphs out of the category of *ranked sets*. This is interesting to me in part because the method doesn't "bake in" the signature of the underlying hypergraphs to the category being considered. This set-up seems considerably more flexible than the one we use in game comonads and understanding it may help to incorporate natural notions of finite model theory such as interpretations into our framework.

Whether these applications turn out to be related directly to the theory of game comonads is not yet known but I believe that understanding them is important to seeing game comonads in their proper context, as part of a wider effort to understand problems previously thought of as combinatorial or algorithmic in a systematic way. Furthermore, these applications all emerging at the same time reinforces the idea that the time is ripe for categorical methods in finite model theory and makes the field very exciting to work in. These are horizons (or co-horizons) I will be watching keenly as I continue with my proposed work.

[4]The original [15] states that any property of finite graphs which is definable in monadic second-order logic with counting is *recognised* by a finitary algebra (which is roughly a generalisation of recognisability by a finite automaton and has as a consequence that the property is decidable in linear time on the class of graphs with bounded tree width). The generalisation presented in [10] is an analogous result on finite hypergraphs

2 Preliminary Research

As demonstrated above in the literature review, the comonadic approach to game-theoretic arguments in finite model theory is a rather recent development with only a small number of papers written about it so far. As a result, the focus of my year has been to familiarise myself with this new technique through a number of smaller research projects. Broadly speaking, these projects divide into three categories, each reflecting an aspect of the work that I plan to do going forward. These categories are:

- *Using comonads to interpret and generalise classical results in finite model theory.*
- *Asking and answering new questions inspired by the comonadic approach.*
- *Relating game comonads with other category-theoretic approaches to finite model theory.*

The three projects I've followed throughout the year which correspond to these categories are, respectively:

- Interpreting preservation theorems with comonads
- Cores in the coKleisli category of a game comonad
- Relating the quantum monad to game comonads

In this section of my report I'll summarise the work I have done in these tracks and the lessons I've learned along the way. I believe the groundwork done in these areas has been fruitful both in what it has revealed about the power and shortcomings of the current formulation of game comonads and for the minor results found along the way. Full sets of notes from this research are available on request.

2.1 Interpreting Preservation Theorems with Comonads

2.1.1 Motivation & Aims

Preservation theorems are central to traditional model theory and they all take something resembling the following general form, for some type of function X between relational structures (homomorphisms, surjective homomorphisms, embeddings) and some fragment $\mathcal{L}_X$ of a logic $\mathcal{L}$ (usually FO):

> If $\phi \in \mathcal{L}$ defines a property $\mathcal{P}$ which is preserved by all maps of type X between all structures, then ϕ is equivalent (over all structures) to some $\phi_X \in \mathcal{L}_X$

Some of the famous theorems of this form in classical model theory are collected in Table 1. The proofs of these often rely on the Completeness Theorem which, as noted in my literature review, fails over finite models. This would

Name	Maps of type X	$\mathcal{L}_X$	Holds in finite?
Łoś-Tarski	Embeddings	$\exists$FO	✗
Lyndon	Surjective Homomorphisms	+FO	✗
Homomorphism Preservation	Homomorphisms	$\exists^+$FO	✓ [33]
Guarded Invariance [5]	Guarded Bisimulation	GF	✓ [32]

Table 1: Some preservation theorems in classical model theory

suggest that there is little hope of finding theorems of the above form when we replace *"all structures"* with *"finite structures"*. However, despite the definite failure of some of these results, there has been some recent success in proving preservation results in the finite. The two examples of this that we chose to study were Rossman's Finite Homomorphism Preservation Theorem [33] and Martin Otto's finite version [32] of Andréka, Németi and van Benthem's Guarded Invariance Theorem[5]. Both have, in contrast to their infinite counterparts, proofs which are difficult and quite involved. In addition, the proofs of these two results are very different from one another, making it satisfying to study them comparatively. For example, Rossman's relies on intricate combinatorial constructions while Otto uses a construction motivated by group theory. We aimed to express these results and their proofs using the comonads of Abramsky et al. and see if any links could be made between these results. We worked on this in collaboration with Samson Abramsky and his student Tom Paine, whose comments and notes were much appreciated.

2.1.2 Summary of Work Done

The approach to this project consisted of two part. In the first part, we looked at Rossman's Equirank Homomorphism Preservation Theorem. Using an unpublished note [3] by Abramsky which expresses this result in terms of the $\mathbb{E}_k$ comonad, we sought to clarify to what extent this approach can be used with other comonads. In particular, we looked at using the $\mathbb{T}_k$ comonad to prove an *equivariable* homomorphism preservation theorem. In the second part, I studied Otto's proof of the Finite Guarded Invariance Theorem. The aim here was to use the systematic group theory of Otto's construction to find an alternative characterisation of the inspired (if somewhat mystical) combinatorial construction that Rossman uses to prove his Finite Homomorphism Preservation Theorem in [33]. The aim was to use the comonadic approach to *extract* the methods used in Otto's work and see what they correspond to in the context of Rossman's work.

Each of these parts led to a short set of rough notes which I hope can be developed further when later work towards my PhD clarifies some of the issues we encountered. Here I will give a brief summary of what we found out.

Towards an equivariable homomorphism preservation theorem Rossman states the equirank homomorphism theorem as follows:

Theorem 3 (Equirank H.P.T)**.** *A first-order sentence is preserved by homomorphisms on all structures if and only if it is equivalent to an existential-positive of equal quantifier rank*

In Rossman's proof of this theorem, he defines a notion of a k*-extendable* structure, which are ones for which $\rightleftarrows_k$ implies $\equiv_k$. The crucial use of this idea is to find a k-extendable cover for all structures. Abramsky rephrases this notion using the $\mathbb{E}_k$ comonad and derives an analogous k-extendable cover, by giving a category-theoretic proof of Rossman's key lemma which classifies k-extendability. This set-up made it natural to ask what happens if the $\mathbb{E}_k$ comonad is replaced with the $\mathbb{T}_k$ comonad, i.e. does the same proof give a proof of an analogous equivariable homomorphism preservation theorem. We formulate this question as follows:

Conjecture 1. *A first-order sentence is preserved by homomorphisms on all structures if and only if it is equivalent to an existential-positive in the same number of variables*

We worked on the analogous idea of k-extendability in the equivariable case for a number of weeks alongside Abramsky and Paine but a proposed proof of this conjecture was seen to fail. This issue has not yet been resolved and seems to hint to get at a subtle difference between $\mathbb{T}_k$ and $\mathbb{E}_k$.

Functorializing Otto Independent of this shared work with Abramsky and Paine, I also undertook to compare Rossman's Finite Homomorphism Preservation Theorem to the Finite Guarded Invariance Theorem proved by Otto in [32]. It was suggested, by Anuj, that the algebraic flavour of the latter could help to come up with a more succinct proof of the former (which takes up the majority of Rossman's original paper and introduces a few dense ad-hoc combinatorial definitions). In my analysis I found that the two proofs are (perhaps necessarily) based on a similar structure, namely, that of finding for each structure A, a finite *extendable* cover $\tilde{A}$. In both cases, extendable means something like $\tilde{A} \rightleftarrows_{k'} B \implies \tilde{A} \equiv_k B$ where for Rossman $\rightleftarrows_{k'}$ means k'-homomorphism equivalence and for Otto it means k' variable bisimulation equivalence. The methods used diverge in how they construct $\tilde{A}$. Rossman tackles this directly by carefully *gluing* together different *cores* of A and showing that for large enough k' the extendability property can be achieved. Otto on the other hand presents a series of reductions, firstly showing that it suffices to construct a highly acyclic hypergraph cover of the hypergraph of A and then that this is equivalent to the existence of certain highly acyclic groups. To transfer this method to the setting of k-homomorphism equivalence, we would like to show what these reductions mean in terms of a guarded bisimulation comonad over A and see if this suggests reductions in the homomorphism equivalence setting. From my work on this, I think it is not too hard to construct the required guarded bisimulation comonad but the difficulty will be in formulating the reductions in this framework. This will likely require a notion of expressing logical interpretations in the comonadic framework, something which I identify as an aim of my research in Section 3.

2.1.3 Conclusions

The conclusions to this section of work were as follows:

- Rossman's proof of the Equirank Homomorphism Preservation Theorem has a nice formulation in terms of the $\mathbb{E}_k$ comonad.
- Proving an equivariable homomorphism preservation theorem turned out not to be as easy as expected and this seemed to be related to a subtle but fundamental difference between the $\mathbb{E}_k$ and $\mathbb{T}_k$ comonads.
- Otto's result, despite it's group theoretic appearance, appears to have more in common with Rossman's result than it first seemed.
- Comparing this in our framework would require developing a notion of a $\mathbb{G}_k$ comonad corresponding to k-variable guarded bisimulation and a notion of expressing logical interpretations in the comonadic framework.

2.2 Cores in the coKleisli Category of a Game Comonad

2.2.1 Motivation & Aims

Central to Rossman's proof of the finite Homomorphism Preservation Theorem (see above) is a notion of the *k-core* of a structure, which is, in essence, a minimal retract of structure which retains satisfies the same $\mathcal{L}^k$ sentences. In our previous project we encountered this idea of *k-core* as a difficulty in generalising the proof of the equirank Homomorphism Preservation Theorem to one for an equivariable Homomorphism Preservation Theorem It seemed that the notion of a "k-variable-core" was not as well-behaved as the equivalent for quantifier rank. We wanted to know why this was the case and so sought to develop a general idea of a *core* of a finite structure over a logic using the comonadic framework of Abramsky, Dawar and Wang. To this end I started by looking at a paper of Hell and Nešetřil [24], which formulates the notion of a core in the category of finite graphs, and I sought to generalise this and apply it to the coKleisli category of the $\mathbb{T}_k$ comonad in the hope of gaining a notion of a k-variable-core.

2.2.2 Summary of Work Done

Firstly, I used Hell and Nešetřil's definition of a graph core to formulate conditions for an arbitrary category of relational structures to have cores. I then proceeded to test these conditions in the coKleisli category $\mathcal{K}(\mathbb{T}_k)$. To do this I needed to characterise the monic and epic maps of $\mathcal{K}(\mathbb{T}_k)$ which I showed to be equivalent to injective and surjective strategies for the k-pebble homomorphism game. Furthermore, I derived a necessary and sufficient logical conditions for such maps between structures A and B to exist in the following lemma

Lemma 1.

1. *If* $A \rightarrow_k^i B \iff A \preceq_{\exists^+\mathcal{C}^k} B$
2. *If* $A \rightarrow_k^s B \iff A \preceq_{+\mathcal{L}^k} B$

To prove this, I adapted Hella's proof of a similar result for bijection games with some subtle changes (one of which I'm particularly proud was an application Hall's Marriage Theorem).

With this characterisation I was able to show that $\mathcal{K}(\mathbb{T}_k)$ satisfied all bar one of the properties mentioned at the start. This meant that $\mathcal{K}(\mathbb{T}_k)$ had a definable notion of a core but that these were not necessarily unique. The final condition states that for any two cores C_1, C_2 of A there should be surjective maps $C_1 \rightarrow_k^s C_2$, $C_2 \rightarrow_k^s C_1$. The latter part of my work on this project focused on this property. I still do not know if this holds or not but in one interesting line of investigation I showed that $\mathcal{K}(\mathbb{T}_k)$ being a regular category would guarantee the property. Then I showed that $\mathcal{K}(\mathbb{T}_k)$ is not regular. Despite this failure, this led me to investigate regular categories and their relationship to logic which I did through reading and writing notes on Carsten Butz's article [13]. Most striking seemed to be what he had to say on the relation between these categories and "Regular Logic", which appears to be the fragment of FO logic that model theorists would call *primitive-positive* first-order logic. As the set of pp-sentences closed under disjunction gives $\exists^+$FO (the fragment of FO which is preserved under homomorphism) it seems natural to ask whether this property of a category of relational structures being a regular category corresponds to the presence of a homomorphism preservation theorem. This might relate this work to Section 2.1, i.e. maybe the failure of regularity for $\mathcal{K}(\mathbb{T}_k)$ can be used to show a failure for an equivariable homomorphism preservation theorem This is the direction I'd like to take this work in going forward. To do this I would need to learn a bit more about categorical logic.

2.2.3 Conclusions

- The properties which guarantee the existence of unique graph cores in [24] can be generalised to arbitrary categories of relational structures.
- Most of these properties succeed in $\mathcal{K}(\mathbb{T}_k)$, meaning that this category has cores but whether they are unique is still open.
- We can characterise the monic and epic maps in $\mathcal{K}(\mathbb{T}_k)$ both game-theoretically and logically.
- Uniqueness of cores would be guaranteed by regularity of $\mathcal{K}(\mathbb{T}_k)$ but this is not the case.

- Regular categories have a connection to a fragment of FO logic this might shine light on the categorical origins of homomorphism preservation theorems.

2.3 Relating the Quantum Monad to Game Comonads

2.3.1 Motivation & Aims

Originally, this project had had the aim of extending the pebbling comonad to give a description of the pebble games with algebraic rules introduced by Dawar and Holm in [17]. These games, as outlined in the literature review, capture a stronger fragment of infinitary FO than the standard k pebble games. They do this by imposing extra algebraic rules on the possible moves of duplicator, making it harder to find a winning strategy. What we are interested in doing is understanding how these rules can be modelled in the language of comonads and category theory. To start off towards this ambitious goal, I sought to get inspiration by investigating other category theoretic methods in finite model theory which use linear algebra in some form. The main candidate for this was the Quantum Monad of Abramsky et al. [1]. This monad $\mathbf{Q}_d$ is defined by sending a relational structure A to a structure on distributions of d-dimensional linear algebraic projectors over the elements of A and it captures the notion of *quantum homomorphism* between relational structures as its Kleisli morphisms. We decided to follow up on an open question posed at the end of [1] about the relationship between this and the pebbling comonad of [2], which asks if there a distributive law between the $\mathbb{T}_k$ comonad and the $\mathbf{Q}_d$ monad. Answering this would allow us to define a biKleisli category for this monad-comonad pair which would give us a notion of a *quantum* winning strategy for the k pebble game. This question is very interesting in its own right but also raised some questions about including algebraic rules in the comonadic finite model theory framework.

2.3.2 Summary of Work Done

The main work so far on this project has involved finding and testing lots of different candidate distributive laws $\lambda : \mathbb{T}_k\mathbf{Q}_d \to \mathbf{Q}_d\mathbb{T}_k$. Given the complexity of the definitions of these (co)monads, finding something to pass even a basic type-check proved quite difficult and we still don't know if there is such a natural transformation (let alone a distributive law). I found some promising candidates however which are given in detail in the notes for this project which are available on request. In doing so, there seemed to be an underlying tension between three properties that λ needed to have, namely normalisation, "forbidding" tuples $(t_1, \dots t_n) \in (\mathbb{T}_k A)^n$ which are "unordered" and "forbidding" tuples $(t_1, \dots t_n) \in (\mathbb{T}_k A)^n$ which are "unrelated" in A. I could find candidates satisfying any two of these rules but not all three. For example, I came up with a technique of *padding* projectors which allowed us to achieve normalisation and forbidding unordered tuples. This technique still needs some work (I found for example that it didn't preserve the property of being a projector). This difficulty (and many others) makes me suspect that in fact there is no distributive law between this pair. To this end I intend to look into some recent No-Go Theorems for distributive laws (of monads over monads) due to Zwart and Marsden in [36]. In future I hope to adapt these to monad-comonad pairs and see if we can disprove that $\mathbb{T}_k$ distributes over $\mathbf{Q}_d$.

2.3.3 Conclusions

- Distributive laws for the pair $(\mathbb{T}_k, \mathbf{Q}_d)$ are hard to find!
- Finding a distributive law in this case (if it exists) will require reconciling the three "competing" properties named above and in my notes.
- Zwart and Marsden's No-Go Theorems are not directly applicable in this case but might be adapted to show that no distributive law can exist here.

3 Description of Proposed Research

As we've seen in this report, this is an exciting time for finite model theory and descriptive complexity. On one hand, there has been surprising progress on long-standing problems. For example,

- A new family of logics, $\mathbf{LA}^k(\Omega)$[16], augmented with linear algebraic operators, and their corresponding games, the k, Ω-IM games[26] have emerged with no known properties in P which they can't decide.
- The Feder-Vardi Dichotomy Conjecture[21], has been resolved, independently by Bulatov [12] and Zhuk[35]
- New preservation theorems, for example due to Rossman[33] and Otto [32] have been shown to hold over finite models

On the other hand, a new research programme initiated by Abramsky and Dawar has begun to recast the fundamental parts of finite model theory in terms of category theory. In their language, the two-way and one-way games used to isomorphism and homomorphism of structures, can be seen as being governed by resources (advantages given to duplicator, e.g. qubits in [1]) and coresources (advantages given to spoiler, e.g. number of pebbles in [2]). In turn, these resources and coresources can be modelled in category theory by graded monads such as $\mathbf{Q}_d$ and comonads, such as $\mathbb{T}_k$. This view has led to fascinating and unexpected insights for finite model theory and hints at a deep new way of thinking about both computational and logical (co)resources in a common framework and the beginning of a new era in descriptive complexity.

My hypothesis is that the recent advances in linear algebraic logic, dichotomy for CSPs and preservation theorems can be interpreted in the categorical language of resource and coresource and that doing so will help us to understand, connect and generalise these results and to ask, and answer, new questions about the deep connection between logic and computation. Testing and developing such a hypothesis in its entirety may constitute an entire research programme rather than the work of a single PhD. However, in pursuit of my PhD, I intend to use this hypothesis a guide to test and push the boundaries of Abramsky and Dawar's theory of resource and coresource.

As this is a newly developing field and it is not yet clear how it will develop, there is a certain amount of risk in pursing this research. In order to mitigate this risk, I will lay out in the next section how I intend to test the ambitious hypothesis described above. I have done this by first dividing the work into two natural tracks: *A coresource theory of the IM game* and *Coresources, cores and preservation theorems.* These are then broken down into a series of interrelated questions and milestones. In my view, and the view of my supervisor, this plan provides a reasonably certain path to a thesis, minimising the risks of working in a relatively new field, while at the same time maintaining a coherent purpose.

4 Timetable & Milestones

In this section, I will lay out more concretely how I intend to progress with the programme of research described previously. The research proposed is divided here for the purposes of clarity into two tracks:

1. A coresource theory of the IM game

2. Coresources, cores and preservation theorems

These tracks are intended to be pursued in tandem and I will interleave work on questions from each track over the next two years to ensure steady progress towards a thesis.

The questions and milestones outlined are intended to take 3-6 months of work each, given any prerequisite work. Not all of the questions below will need to be answered to constitute a substantial contribution to the emerging theory of resource and coresource. In my opinion and that of my supervisor, a collection of results from both tracks should make a compelling and coherent thesis.

Diagrams are provided to indicate what Prof. Dawar and I think is a logical way to approach the questions arising in each of these areas. The full arrows ($A \longrightarrow B$) denote that resolving A should enable B (or at least the intention to do A after B) and the dotted lines ($A \cdots\cdots B$) indicate relation or similarity between A and B. In many cases however, as with all research, the set of arrows presented and their direction is only indicative and things may change as I learn more about these questions and pose new ones.

4.1 A coresource theory of the IM game

One of the major insights provided by the first paper on the pebbling comonad was the connection it provided between the approximation to homomorphism provided by the k-local consistency method for CSPs (also seen in the one-way k pebble game) and the $(k-1)$-Weisfeiler-Lehman approximation to isomorphism (also the k pebble bijective game).

This track of research aims to investigate if coresource theory will reveal relationships between stronger approximations to homomorphism, such as k-local consistency augmented with the few subpowers algorithm of Barto and Kozic, and stronger approximations to isomorphism, such as the k-invertible maps (k-IM) equivalence. This track intends to answer this question by developing a coresource theory of the IM game. The main goal in doing so will be to develop a tool similar to the comonad of Abramsky, Dawar and Wang in this richer context. I provide below a roadmap of how I will work towards this aim.

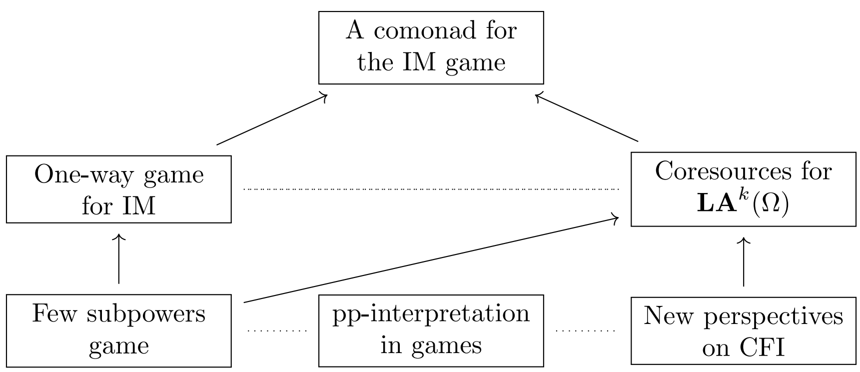

Figure 1: Towards a (co)resource view of the IM game

1. **A game for few subpowers** To get a one-way version of the k-IM game it might help to look at other approximations to homomorphism of relational structures with a similar linear algebraic structure. One such is the few subpowers algorithm [9] which generalises Gaussian elimination and represents (along with k-local consistency) one of the two P algorithms for CSP.

   Question: Is there a one-way game where duplicator wins from A to B if and only if the few subpowers algorithm accepts (A, B)?

2. **pp-interpreting in games** The work of Atserias Bulatov and Dawar [6] showed that relational structures which can pp-interpret modular arithmetic, admit CFI-like constructions which cannot be distinguished in $C^k_{\infty\omega}$. This currently is not accounted for in the compositional framework. I'd like to do so by relating the $\mathbb{T}_k$ comonad on one signature to

the $\mathbb{T}_k$ comonad on another signature which can be pp-interpreted by the original.

> *Question:* If σ pp-interprets σ' what is the relationship between $\mathbb{T}_k^{\sigma}$ and $\mathbb{T}_k^{\sigma'}$?

3. **New perspectives on CFI** To understand the resource limitation that's taking place in the k-IM game and in looking at finite structures up to $\mathbf{LA}^k(\Omega)$-equivalence, it will help to understand the extra expressive power of these. This extra expressive power is usually understood through the graphs of Cai-Furer-Immerman [14] which are not distinguishable in $C^k_{\infty\omega}$ but are distinguished by k-IM games. These graphs are still not widely understood. An idea for understanding this might be to look at another situation in which computation of rank over finite fields is relevant. Simplicial (co)homology is one such idea but there may be others.

   > *Question:* Is there a query on graphs (perhaps inspired by topology) which is expressible in IFP+rk but not in IFP+C?

4. **A one-way game for IM** The standard strategy for building the coresource theory of a given game in finite model theory has been to encode a one-way game into a comonad so that the coKleisli morphisms are exactly the approximation to homomorphism in this resource-limited view. Doing this for IM games is hard because the game is baked in at multiple levels to be bidirectional. Picking this apart will hopefully reveal more about the resources being limited in this game.

   > *Question:* Can we weaken the rules and restrict spoiler sufficiently in the k, Ω-IM game to get a truly one-way version of this game?

5. **Coresources for $\mathbf{LA}^k(\Omega)$** A one way game is one way to find a compositional framework for the IM game but it may not be the only one. Perhaps a deeper understanding of other ways in which linear algebra features in approximating homomorphism and isomorphism will give us a more direct "coresource" with which to model the k-IM game.

   > *Question:* Is there a single parameter or syntactic feature of $\mathbf{LA}^k(\Omega)$ sentences that we can limit to get an approximation of homomorphism? If not how does we restrict both variables and the primes of Ω work?

6. **A comonad for IM game** Having a compositional framework to understand the k-IM game of Dawar and Holm [17] would be a big achievement for the theory of resource and coresource and would represent uniting the state-of-the-art of descriptive complexity with this category theoretic view.

   > *Question:* Is there a comonad (or other resource-restricting structure) capturing the k, Ω-IM game?

4.2 Coresources, cores and preservation theorems

Preservation theorems have always been of interest to finite model theorists. Ever since Trakhtenbrot showed the failure of completeness over finite models and thus the failure of many standard proofs of preservation theorems in classical model theory, finite model theorists have asked which of these are just untrue in the finite and which are redeemable. The result is an interesting patchwork; some fail (such as Los-Tarski and Lyndon's Positivity Theorem), some succeed often with rather formidable proofs (such as the Homomorphism Preservation and Bisimulation and Guarded Bisimulation Preservation) and some, such as the Janin-Walukiewicz Theorem, are not known to fail or succeed.

This track of work aims to build on the investigations I've undertaken this year on cores and preservation theorems which indicates that the theory of resource and coresource will help to explain this patchwork and may offer new proofs of old results and may even resolve some unanswered questions about preservation over finite models. To get to a general coresource theory of preservation theorems I will build on my work and partial results from this year while also extending the coresource theory to cover various notions of bisimulation for which there exist a variety of preservation theorems and open questions. Overall, I would hope to find a way of explaining which types of resource limitations lead to preservation theorems and which lead to failure of classical results.

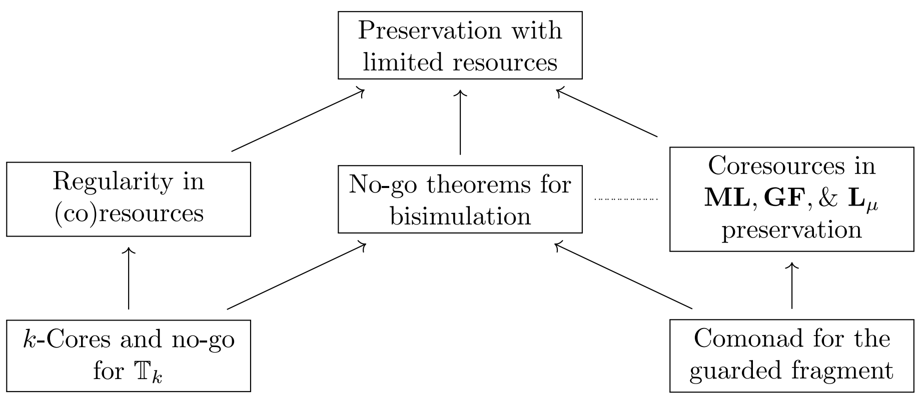

Figure 2: Towards a (co)resource view of preservation theorems

1. **k-Cores and no-go theorems for** $\mathbb{T}_k$ My exploration of k-cores so far has led to some interesting failures of standard constructions (namely cores[24]) when our model theory is resource limited to k variables. It is believed that this is connected to the no-go theorem about the impossibility of finding a finitary version of $\mathbb{T}_k$ proven in [2]

   Question: For $\mathbb{T}_k$ (of any game comonad $\mathbb{G}_k$) does a no-go theorem for a finitary version of the comonad imply the nonexistence of cores in its coKleisli category, and vice versa?

2. **A comonad for guarded bisimulation** Guarded bisimulation is a limited bisimulation relation between hypergraphs of relational structures which was shown by Otto in [32] to have an interesting FMT preservation theory. No comonad is known for this bisimulation yet. Finding one would be a contribution to the field and would allow us to consider what features of this comonad are necessary for Otto's proof.

   > *Question:* Is there a game comonad $\mathbb{GF}_k$ for which the isomorphisms of its coKleisli category correspond to a resource limited form of guarded bisimulation?

3. **Regularity in (co)resources** In my work on cores, I came across a curious sufficient condition for categories to permit cores, namely regularity. Regular categories also seem deeply related in categorical logic (see for example Butz [13]) to regular logic which is know to finite model theorists as primitive-positive first-order logic and is central to much of FMT.

   > *Question:* Can coresource comonads with non-regular coKleisli categories admit preservation theorems?

4. **No-go theorems for bisimulation** In reasoning about pebble games, we often think of them using the same imagery of infinite tree unfoldings as is often used to describe bisimulation. On way of thinking of Abramsky, Dawar and Wang's No-Go Theorem for the $\mathbb{T}_k$ comonad is that this unfolding for pebble games is necessarily infinite in some way. One way to think of Otto's result is that for guarded bisimulation, the unfolding can be finitised. I think it will be important to understanding preservation theorems to understand which bisimulation comonads have this necessary infiniteness and which do not.

   > *Question:* Does the $\mathbb{GF}_k$ comonad (but also do comonads for **ML** and $\mathbf{L}_\mu$) admit no-go theorems/cores/regular coKleisli categories?

5. **Coresources in $\mathbf{ML}, \mathbf{GF}, \&\ \mathbf{L}_\mu$ preservation** There are finite bisimulation preservation theorems for both **ML**, due to Rosen, and **GF**, due to Otto. There is a similar result for modal μ-calculus, $\mathbf{L}_\mu$, which is not yet know to hold in the finite. With a comonad for guarded bisimulation and one modal logic (which exists in [4]), I think it would be an interesting test of the theory of coresources to develop a comonad for $\mathbf{L}_\mu$ and investigate whether there is a coresource-based explanation for the differences in preservation theorems for these.

   > *Question:* Can we create a game comonad for $\mathbf{L}_\mu$? Does comparing this to the comonads for **ML** and **GF** explain why there are preservation theorems for two but not yet for the third?

6. **Preservation theorems in resource-limited views** Rossman and Otto have proved preservation theorems recently that were believed by some not to hold over finite models. My work on cores has shown that in some resource limited views some of these may not hold. What differentiates these settings is an interesting line of study and giving a category-theoretic account of it would be a major achievement.

   Question: Is there a general theorem governing when a resource-limited view will admit a preservation theorem?

References

[1] Abramsky, S., Barbosa, R. S., de Silva, N., and Zapata, O. The Quantum Monad on Relational Structures. In *42nd International Symposium on Mathematical Foundations of Computer Science (MFCS 2017)* (Dagstuhl, Germany, 2017), K. G. Larsen, H. L. Bodlaender, and J.-F. Raskin, Eds., vol. 83 of *Leibniz International Proceedings in Informatics (LIPIcs)*, Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, pp. 35:1–35:19.

[2] Abramsky, S., Dawar, A., and Wang, P. The pebbling comonad in finite model theory. In *2017 32nd Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)* (June 2017), pp. 1–12.

[3] Abramsky, S., and Paine, T. Functorializing Rossman. Forthcoming.

[4] Abramsky, S., and Shah, N. Relating structure and power: Comonadic semantics for computational resources. *CoRR abs/1806.09031* (2018).

[5] Andréka, H., Németi, I., and van Benthem, J. Modal languages and bounded fragments of predicate logic. *Journal of Philosophical Logic 27* (06 1998).

[6] Atserias, A., Bulatov, A., and Dawar, A. Affine systems of equations and counting infinitary logic. *Theoretical Computer Science 410*, 18 (2009), 1666 – 1683. Automata, Languages and Programming (ICALP 2007).

[7] Babai, L. Graph isomorphism in quasipolynomial time. *CoRR abs/1512.03547* (2015).

[8] Baldwin, J. T. Finite and infinite model theory - a historical perspective. *Logic Journal of the IGPL 8* (2000), 605–628.

[9] Barto, L., and Kozik, M. Absorption in Universal Algebra and CSP. In *The Constraint Satisfaction Problem: Complexity and Approximability*, A. Krokhin and S. Zivny, Eds., vol. 7 of *Dagstuhl Follow-Ups*. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, Dagstuhl, Germany, 2017, pp. 45–77.

[10] Bojańczyk, M. Two monads for graphs. *CoRR abs/1804.09408* (2018).

[11] Bulatov, A., Jeavons, P., and Krokhin, A. Classifying the complexity of constraints using finite algebras. *SIAM J. Comput. 34*, 3 (Mar. 2005), 720–742.

[12] Bulatov, A. A. A dichotomy theorem for nonuniform CSPs. *CoRR abs/1703.03021* (2017).

[13] Butz, C. Regular categories and regular logic. *BRICS Lecture Series LS-98-2* (1997).

[14] Cai, J.-Y., Fürer, M., and Immerman, N. An optimal lower bound on the number of variables for graph identification. *Combinatorica 12*, 4 (Dec 1992), 389–410.

[15] Courcelle, B. The monadic second-order logic of graphs. I. recognizable sets of finite graphs. *Information and Computation 85*, 1 (1990), 12 – 75.

[16] Dawar, A., Grädel, E., and Pakusa, W. Approximations of isomorphism and logics with linear-algebraic operators. *CoRR abs/1902.06648* (2019).

[17] Dawar, A., Grohe, M., Holm, B., and Laubner, B. Logics with rank operators. In *Proceedings of the 2009 24th Annual IEEE Symposium on Logic In Computer Science* (Washington, DC, USA, 2009), LICS '09, IEEE Computer Society, pp. 113–122.

[18] Dawar, A., and Holm, B. Pebble games with algebraic rules. *Fundamenta Informaticae Vol. 150, nr 3/4* (2017), 281–316.

[19] Ehrenfeucht, A. An application of games to the completeness problem for formalized theories. *Fundamenta Mathematicae 49*, 2 (1961), 129–141.

[20] Fagin, R. *Contributions to the Model Theory of Finite Structures.* PhD thesis, University of California at Berkeley, 1973.

[21] Feder, T., and Vardi, M. The computational structure of monotone monadic SNP and constraint satisfaction: A study through datalog and group theory. *SIAM J. Comput. 28* (01 1998), 57–104.

[22] Grädel, E., and Pakusa, W. Rank logic is dead, long live rank logic! *The Journal of Symbolic Logic 84*, 1 (2019), 54–87.

[23] Grohe, M. Finite variable logics in descriptive complexity theory. *The Bulletin of Symbolic Logic 4*, 4 (1998), 345–398.

[24] Hell, P., and Nešetřil, J. The core of a graph. *Discrete Mathematics 109*, 1 (1992), 117 – 126.

[25] Hella, L. Logical hierarchies in PTIME. *Information and Computation 129*, 1 (1996), 1 – 19.

[26] Holm, B. *Descriptive complexity of linear algebra.* PhD thesis, University of Cambridge, UK, 2011.

[27] Immerman, N. Relational queries computable in polynomial time. *Information and Control 68*, 1 (1986), 86 – 104.

[28] Immerman, N., and Lander, E. *Describing Graphs: A First-Order Approach to Graph Canonization.* Springer New York, New York, NY, 1990, pp. 59–81.

[29] Jeavons, P., Cohen, D., and Gyssens, M. Closure properties of constraints. *J. ACM 44*, 4 (July 1997), 527–548.

[30] Kolaitis, P. G., and Vardi, M. Y. Infinitary logic for computer science. In *Automata, Languages and Programming* (Berlin, Heidelberg, 1992), W. Kuich, Ed., Springer Berlin Heidelberg, pp. 450–473.

[31] Libkin, L. *Elements Of Finite Model Theory (Texts in Theoretical Computer Science. An Eatcs Series)*. SpringerVerlag, 2004.

[32] Otto, M. Highly acyclic groups, hypergraph covers, and the guarded fragment. *J. ACM 59*, 1 (Mar. 2012), 5:1–5:40.

[33] Rossman, B. Homomorphism preservation theorems. *J. ACM 55*, 3 (Aug. 2008), 15:1–15:53.

[34] Vardi, M. Y. The complexity of relational query languages (extended abstract). In *Proceedings of the Fourteenth Annual ACM Symposium on Theory of Computing* (New York, NY, USA, 1982), STOC '82, ACM, pp. 137–146.

[35] Zhuk, D. The proof of CSP dichotomy conjecture. *CoRR abs/1704.01914* (2017).

[36] Zwart, M., and Marsden, D. Don't Try This at Home: No-Go Theorems for Distributive Laws. *arXiv e-prints* (Nov 2018), arXiv:1811.06460.