Exercises for the Lectures on

Bigraphs: a Model for Mobile Agents

Exercises for Lecture I

E1 DECOMPOSITION. Recall the convention that we write the inner names of a bigraph below the regions that represent it.

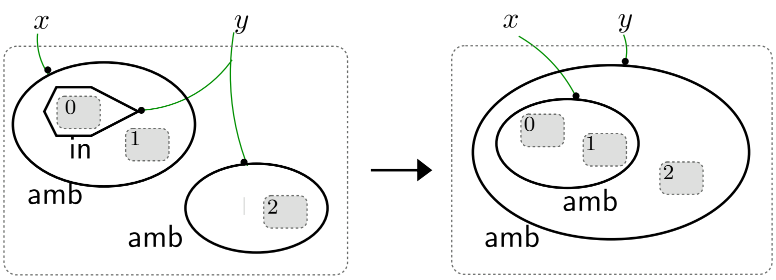
For the built environment G, draw a bigraph D representing the three agents that are inside rooms, and a host bigraph F such that $G = F \circ D$. Write the interface between F and D in algebraic form.

E2 DESIGNING RULES.
(a) The set B1–B3 of reaction rules is very limited. To extend it a little, add a rule B4 to enable an agent linked with a computer to sever this link, and another rule B5 to allow an agent unlinked to a computer to leave a room.

(b) Given a set of reaction rules, we call a property of a bigraph B an *invariant* if it holds of all possible future states. An obvious invariant of $H \circ G$ is 'there are exactly five agents'. Think of five other invariants of $H \circ G$ under the rules B1–B5.

(c) Instead of B4 and B5, design a single rule B6 that allows an agent to leave a room, simultaneously severing any link with the computer. Does this change affect your invariants? Can you think of another which did not hold before?

E3 MOBILE AMBIENTS. Here is one of the reaction rules for Mobile Ambients, a calculus which inspired the treatment of placing in bigraphs.



$$\mathsf{amb}_x.(\mathsf{in}_y.d_0 \mid d_1) \mid \mathsf{amb}_y.d_2 \longrightarrow \mathsf{amb}_y.(\mathsf{amb}_x.(d_0 \mid d_1) \mid d_2)$$

The in-node, inside an ambient named x and pointing to an ambient named y, causes the former ambient to move *inside* the latter one. At the same time the in-node dies, but its contents are released.

We have not yet introduced the algebra of bigraphs, but in this instance it explains itself; note that the contents of sites $0, 1 \ldots$ are denoted by $d_0, d_1, \ldots$.

Conversely, suppose that we have an ambient named x inside an ambient named y and we want the former to move *outside* the latter. Using a new control 'out', analogous to ‘in’, design a rule which will make this happen when there is an out-node inside the x-ambient. Write down the algebraic expression of this rule.

Finally, introduce a new control ‘throw’. Design a rule such that a throw-node, pointing to an ambient named x, will ‘throw’ its contents into that ambient and then die. *Hint:* to allow the throwing node to be anywhere outside the ambient it points to, perhaps inside other ambients, use a reaction rule that has two regions (unlike the rules we have met so far).

Exercises for Lecture II

E4 BIGRAPHS AS A CATEGORY. Assume that the identity bigraphs are defined by $\mathsf{id}_{\langle m,X\rangle} \stackrel{\text{def}}{=} \langle \mathsf{id}_m, \mathsf{id}_X\rangle$ and composition of bigraphs by $G \circ F \stackrel{\text{def}}{=} \langle G^{\mathsf{P}} \circ F^{\mathsf{P}}, G^{\mathsf{L}} \circ F^{\mathsf{L}}\rangle$. Assuming also that place graphs and links graphs both form categories, prove that bigraphs form a category.

E5 ORIGIN. Given a signature $\mathcal{K}$, what bigraphs have the origin ϵ as their outer face? Which of these have empty support?

E6 LINKINGS. Show that every linking can be built from elementary linkings using identities, composition and product. Is composition necessary for this?

E7 OCCURRENCE. Let us define formally what it means for one bigraph to occur in another. We say that F *occurs* in G if and only if the equation

$$G = C_1 \circ (F \otimes \mathsf{id}_I) \circ C_0$$

holds for some interface I and some bigraphs C_0 and C_1.

The identity id_I is important here: it allows nodes of C_1 to have children in C_0 as well as in F, and allows C_1 and C_0 to share links that do not involve F. It seems to be the natural way to define occurrence, as the following suggests.

Prove that F occurs in G, according to this definition, if G is takes any of the forms $F \circ C$, $C \circ F$, $F \otimes C$ or $C \otimes F$. Also show that a *ground* bigraph a (one whose inner face is ϵ) occurs in a ground bigraph g if and only if $g = C \circ a$ for some C.

Also prove that occurrence is transitive, i.e. if E occurs in F and F occurs in G then E occurs in G.

You will need the equations satisfied by composition and product in an s-category. For simplicity, assume that the compositions and products you use are defined.

E8 STRATIFICATION. Let us say that a place sorting $\Sigma = (\Theta, \mathcal{K}, \Phi)$ is *stratified* if, for some function $\phi : \Theta \to \Theta$, the formation rule Φ requires that

all children of a root $r : \theta$ have sort θ ;
all children of a node $v : \theta$ have sort $\phi(\theta)$.

The CCS place-sorting Σ_{ccs} for CCS is a special case of stratified sorting in which $\Theta = \{\mathsf{p}, \mathsf{a}\}$ (for processes and alternations), with $\phi(\mathsf{p}) = \mathsf{a}$ and $\phi(\mathsf{a}) = \mathsf{p}$.

Prove that every stratified sorting is satisfied by the identities and symmetries, and preserved by composition and product. Why is it necessary for interfaces to contain sorts, in order to achieve this?

E9 DISTRIBUTED CCS. Taking the hint from process calculi with locality, let us make a simple extension of CCS which we could call ‘distributed CCS’. We add an extra top-level syntactic entity called a *system*. A system S consists of a set of *cells*, each containing a single process. We shall allow two forms of communication; one (as we have now) will allow communication only within a cell, and the other will allow communication between processes whether or not in the same cell.

So we extend the syntax class of actions μ, and we add a syntax class of systems, as follows:

$$\begin{array}{lcl} \mu & ::= & \overline{x} \;\big|\; \overline{\overline{x}} \;\big|\; x \\ S & ::= & (\!|P|\!) \;\big|\; S, S \end{array}$$

where $\overline{\overline{x}}$ ‘throws’ an output that can be received anywhere, and $(\!|P|\!)$ is a cell containing the process P.

How should structural congruence be adjusted? What change should be made to the signature $\mathcal{K}_{\mathsf{ccs}}$ and the sorting Σ_{ccs}? What extension must be made to the translation from CCS to $\mathrm{B}\textsc{g}_{\mathsf{ccs}}$? (Changes to the dynamics of $\mathrm{B}\textsc{g}_{\mathsf{ccs}}$ will be the subject of later exercises.)

Exercises for Lecture III

E10 DISTRIBUTED REACTION. Continuing Exercise E9, what extra reaction axioms and rules do we need, beyond those in the slide ‘Reaction in CCS’, to allow communication both within a cell and across cell boundaries? *Hint:* it is not as elegant as you might hope!

E11 ALGEBRA. Write down in algebraic form the ground reaction rule $(r, r') = (R.d, R'.d')$ from the parametric CCS rule shown in the slide ‘Reaction in CCS bigraphs’, assuming a parameter $d = d_0 \otimes \cdots \otimes d_3$, where d_i has outer names Y_i $(0 \le i \le 3)$ and $Y = \biguplus_i Y_i$.

Give the interfaces of R, R', d, r and r', assuming the stratified sorting for CCS.

E12 ACTION AT A DISTANCE. Continuing Exercise E10, we now wish to model thrown reactions in the bigraphs for CCS. What extension is needed to the dynamic signature of $\mathrm{B}\textsc{g}_{\mathsf{ccs}}$?

Adapt the parametric rule of the slide ‘Reaction in CCS bigraphs’ to allow thrown reactions to occur both within and between cells.

E13 BEHAVIOURAL EQUIVALENCE. Consider the two CCS processes

$$P = x.(y.0 + z.0) \text{ and } Q = x.y.0 + x.z.0$$

where x, y and z are distinct channel names. In what sense do they behave the same? In what sense, if any, do they behave differently?

E14 BISIMILARITY. As part of the original proof for CCS that bisimilarity is a congruence, prove that if $P_1 \sim P_2$ then $P_1 \,|\, Q \sim P_2 \,|\, Q$. *Hint:* To do this, define the relation

$$\mathcal{S} \stackrel{\text{def}}{=} \{(P_1 \,|\, Q, P_2 \,|\, Q) \;\big|\; P_1 {\sim} P_2,\ Q \text{ any process}\} \;.$$

To prove it is a bisimulation, consider any transition $P_1 \,|\, Q \xrightarrow{\mu} R_1$; you have to find an R_2 such that $P_2 \,|\, Q \xrightarrow{\mu} R_2$ and $(R_1, R_2) \in \mathcal{S}$. To do this consider all the ways in which the transition of $P_1 \,|\, Q$ could be inferred from the transition rules (for this exercise ignore the rule involving $\equiv$), showing that in each case an appropriate R_2 can be found.

E15 MATCHING A REDEX. The slides called 'What's a minimal bound?' illustrate how the minimal bounds for a pair (a, r) depend on the overlap between a and r. This overlap can be defined by tagging their nodes with identifiers. It is not obvious how many so-called *matches* there are. This exercise is designed to give a flavour of how to answer this question. (For the full treatment, see the discussion of *consistent spans* in the book.)

The slides show two matchings between a and r, and the corresponding bounds (L, D). In the first case, they share their left-hand send-nodes; in this case prove that, to achieve $L \circ a = D \circ r$, they *cannot* share their get-nodes.

In the second case, a shares its left-hand send-node with r's right-hand send-node; in this case prove that, to achieve $L \circ a = D \circ r$, they *must* share their get-nodes.

What other matches between a and r are possible, to achieve $L \circ a = D \circ r$?

Exercises for Lecture IV

E16 BUD FISSION. Our reaction rules for membrane budding deserve a closer look. The rule for bud fission removes the gates that allow particle migration, but the bud remains a bud, rather than becoming a new brane. Therefore it cannot behave like a brane, and create buds itself. Before that it must 'become' a brane.

Without knowing the biological reality, let us assume that the process of bud fission allows the coat proteins to be shed, and after that the bud becomes a fully fledged brane.

Make this happen, perhaps by adding new controls and adding or modifying reaction rules.

E17 PARTICLE MIGRATION. The rule for particle migration allows particles to move in both directions between a brane and an incipient bud. When we assign rates to reaction rules, the rate assigned to this rule will not favour one direction of movement over the other. In reality, it may be that movement of particles into the incipient bud is actually faster than in the other diection.

To model this, we need two rules with different rates. Design these two rules, to replace our present rule for particle migration.

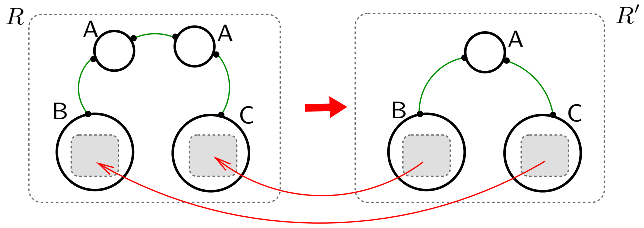
E18 STOCHASTIC SIMULATION. To say that an event has rate ρ means that the probability that it occurs (at or) later than time $t \geq 0$ is $e^{-\rho t}$. Suppose that n reactions are possible in a state g, with rates $\rho_1, \ldots, \rho_n$. Let $\rho = \sum_{i=1}^{n} \rho_i$. Prove that the probability p_i of the i^{th} reaction occurring first is ρ_i/ρ.

This is the basis of a stochastic interpreter. In each state g it discovers the rates and results of the possible reactions, and picks the next state g' non-deterministically, with the probability of the reaction $g \longrightarrow\!\!\triangleright\, g'$ computed as above.

Hint: What is probability that all the reactions except the i^{th} occur at or later than time t? What is the probability that the i^{th} reaction occurs in the infinitesimal time interval $(t, t + \delta t)$? From these two, compute the required result by integration.

Exercises for Lecture V

E 19 TRACKING. Consider a simple parametric rule (R, R', η), in which an A-node is lost and the B- and C-nodes swap their contents.



This description is misleading! Which A-node is lost? Or maybe both are lost and a new one created? Hitherto, in abstract bigraphs, we have asserted nothing about whether nodes keep their identity through a reaction. With tagging, we can do so.

In concrete bigraphs, we can enrich a rule to a quadruple (R, R', η, τ) by adding a partial map $\tau : |R'| \rightharpoonup |R|$ called a *tracking map*. It tells us whether an entity s in R' is newly created (i.e. $\tau(s)$ undefined), or if not, where it can be tracked to in R. Answer several questions:

(a) Is there any reason that a node should not change its control through a reaction?

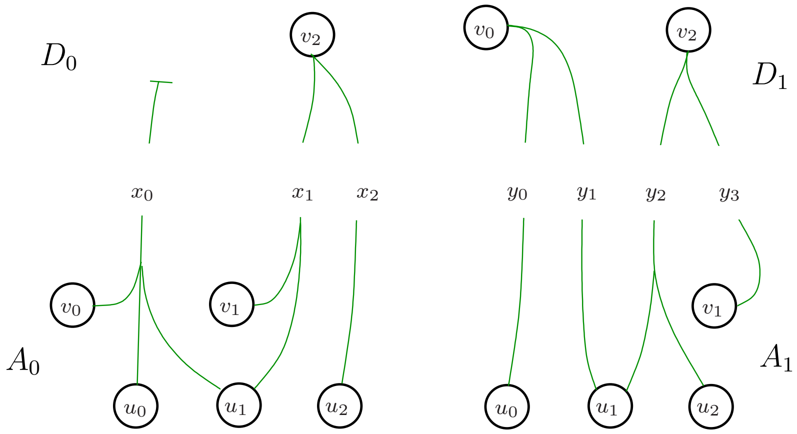
(b) Let (r, r') be a ground rule generated by (R, R', η, τ). How would you extend the tracking map to $|r'| \rightharpoonup |r|$?

(c) If $g \longrightarrow g'$ is a reaction based upon (r, r'), how would you extend the tracking map to $|g'| \rightharpoonup |g|$?

(d) If $g \longrightarrow g_1 \longrightarrow \cdots\cdots \longrightarrow g_n = g'$, how would you define a tracking map $|g'| \rightharpoonup |g|$ for the whole reaction sequence?

(e) Do you think tracking—perhaps generalised—is a way to extract a causal structure for bigraphical dynamics?

E20 A LINK GRAPH RPO. Find the RPO $(\vec{B}, B)$ of the following bound $\vec{D}$ for $\vec{A}$:



E21 RPOS FOR BIGRAPHS. In bigraphs, let $\vec{D}$ be a bound for $\vec{A}$. Then we can construct a place graph RPO $(\vec{B}^{\mathsf{P}}, B^{\mathsf{P}})$ for $\vec{A}^{\mathsf{P}}$ relative to $\vec{D}^{\mathsf{P}}$. We can also construct a link graph RPO $(\vec{B}^{\mathsf{L}}, B^{\mathsf{L}})$ for $\vec{A}^{\mathsf{L}}$ relative to $\vec{D}^{\mathsf{L}}$. It also turns out that the triple $(\vec{B}, B)$, where

$$B_i = \langle B_i^{\mathsf{P}}, B_i^{\mathsf{L}} \rangle \ (i = 0, 1) \ \text{ and } \ B = \langle B^{\mathsf{P}}, B^{\mathsf{L}} \rangle \ ,$$

is well formed, because the node-sets in each pair coincide.

Using this fact, and the properties of combination, prove that $(\vec{B}, B)$ is indeed an RPO for $\vec{A}$ relative to $\vec{D}$.

E22 BEHAVIOURAL CONGRUENCE. Complete the proof of congruence for minimal transitions in two respects:

(1) To ensure that $\mathcal{S}$ is a bisimulation, we require that $(b_0', b_1') \in \mathcal{S}$, i.e. that for some context C' we have $(b_0', b_1') = (C' \circ a_0', C' \circ a_1')$ with $a_0' \sim a_1'$.

To achieve this, first note that (a) is based on a ground rule (r_0, r_0') with $b_0' = E_0 \circ r_0'$; similarly, (c) is based on a ground rule (r_1, r_1') with $a_1' = D_1' \circ r_1'$. Then complete the argument.

(2) Since we are given the transition of $C \circ a_0$ we know that E_0 is an active context. To justify the constructed transition for $C \circ a_1$ we need that E_1 is also active. Prove this claim.

We have also ignored some details concerning support equivalence; these can be found in the book.