Nominal Cubical

model of type theory

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↑

[CCHM]
C. Cohen, T. Coquand,
S. Huber, and A. Mörtberg,
Cubical type theory: a constructive interpretation of the univalence axiom.
ArXiv e-prints, arXiv:1611.02108, 2016.

Nominal Cubical

model of type theory

The CCHM model of Homotopy Type Theory can be reformulated using (some) nominal techniques.

> A. M. Pitts, *Nominal Sets: Names and Symmetry in Computer Science*, Cambridge Tracts in Theoretical Computer Science, vol. 57 (CUP, 2013)

This simplifies the description if some parts of the model and may lead to new models of univalence.

Plan

- Motivation: the univalence axiom [HoTT]
- Overview of the Cohen-Coquand-Huber-Mörtberg presheaf model of univalent type theory [CCHM,OP,B+]
- Toposes of $\mathbb{M}$-sets
- CCHM cubical sets as finitely supported $\mathbb{M}$-sets [Pit]
- Path objects
- Cofibrant propositions and fibrant families
- A univalent universe [CCHM]

Main sources

[HoTT] The Univalent Foundations Program, *Homotopy Type Theory: Univalent Foundations for Mathematics*. Institute for Advanced Study, 2013.

[CCHM] C. Cohen, T. Coquand, S. Huber, and A. Mörtberg, *Cubical type theory: a constructive interpretation of the univalence axiom*. arXiv:1611.02108.

[OP] I. Orton and A. M. Pitts, *Axioms for modelling cubical type theory in a topos*, Proc. CSL 2016.

[B+] L. Birkedal, A. Bizjak, R. Clouston, H. Grathwohl, B. Spitters, A. Vezzosi, *Guarded Cubical Type Theory: Path Equality for Guarded Recursion*, Proc. CSL 2016.

[Pit] A. M. Pitts, *Nominal Presentation of Cubical Sets Models of Type Theory*, Proc. TYPES 2014.

[Nom] A. M. Pitts, *Nominal Sets: Names and Symmetry in Computer Science*, Cambridge Tracts in Theoretical Computer Science, vol. 57 (CUP, 2013).

[Hof] M. Hofmann, *Syntax and semantics of dependent types*. In A.M. Pitts and P. Dybjer (eds), Semantics and Logics of Computation, pp 79–130 (CUP, 1997).

Univalence

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given $X, Y : \mathcal{U}$, every $p : X =_{\mathcal{U}} Y$

type of **identifications** (proofs of equality) between X and Y in $\mathcal{U}$

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given $X, Y : \mathcal{U}$, every $p : X =_{\mathcal{U}} Y$ induces an isomorphism $X \cong Y$ (relative to $=$).

$$p_* : X \to Y \qquad p^* : Y \to X$$
$$\eta : (\mathbf{id} =_{Y \to Y} p_* \circ p^*) \qquad \varepsilon : (p^* \circ p_* =_{X \to X} \mathbf{id})$$

well-defined by just giving the case when $p \equiv \mathbf{refl}$

(for which $p_* \equiv p^* \equiv \lambda x.\, x$ and $\eta \equiv \varepsilon \equiv \mathbf{refl}$)

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$\mathcal{U}$ is **univalent** if there is a proof of "all isomorphisms $X \cong Y$ in $\mathcal{U}$ are induced by some $p : X =_{\mathcal{U}} Y$".
(Notation: UTT $\equiv$ MLTT $+$ univalence.)

Licata, Shulman *et al*: the above is logically equivalent to, but a bit simpler than Voevodsky's original definition.

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N.B. univalence is inconsistent with extensional type theory (ETT).

ETT satisfies: if $p : x =_A y$,
then $x \equiv y$ and $p \equiv \mathbf{refl}$

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Homotopy Type Theory to the rescue: elements $p : x =_A y$ are analogous to paths p from point x to point y in a space A with $\mathbf{refl} : x =_A x$ corresponding to a constant path [Awodey-Warren, Voevodsky,...]

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All (?) existing models with non-truncated univalent universes stem in some way from:

- Kan simplicial sets in classical set theory [Voevodsky *et al*]
- uniform-Kan cubical sets in constructive set theory [CCHM]

(We need more, and simpler, examples!)

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Overview of the CCHM model

Uses Dybjer's Category with Families (CwF) for the semantics of MLTT.

Brief recap here – see [Hof] for details.

Category with Families (CwF)

A CwF is given by

- category $\mathcal{C}$ with a terminal object $\mathbf{1}$
  [objects $\Gamma, \Delta, \ldots \in \mathcal{C}$ model typing contexts;
  morphisms $\gamma \in \mathcal{C}(\Delta, \Gamma)$ model simultaneous substitutions mapping variables to terms (context morphisms);
  $\mathbf{1}$ denotes the empty context]

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- category $\mathcal{C}$ with a terminal object $\mathbf{1}$
- for each $\Gamma \in \mathcal{C}$, a set $\mathcal{C}(\Gamma)$ of families over Γ and for each $\gamma \in \mathcal{C}(\Delta, \Gamma)$ a re-indexing function $\_[\gamma] : \mathcal{C}(\Gamma) \to \mathcal{C}(\Delta)$, functorial in γ

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  [families model types-in-context; re-indexing models substitution of terms for variables in types]

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- for each $\Gamma \in \mathcal{C}$, a set $\mathcal{C}(\Gamma)$ of families over Γ
- for each $\Gamma \in \mathcal{C}$ and $A \in \mathcal{C}(\Gamma)$, a set $\mathcal{C}(\Gamma \vdash A)$ of elements of the family A over Γ
  and for each $\gamma \in \mathcal{C}(\Delta, \Gamma)$ a re-indexing function
  $\_[\gamma] : \mathcal{C}(\Gamma \vdash A) \to \mathcal{C}(\Delta \vdash A[\gamma])$,
  (dependently) functorial in γ

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  (dependently) functorial in γ
  [elements model terms-in-context of a given type; re-indexing models substitution of terms for variables in terms]

Category with Families (CwF)

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- for each $\Gamma \in \mathcal{C}$, a set $\mathcal{C}(\Gamma)$ of families over Γ
- for each $\Gamma \in \mathcal{C}$ and $A \in \mathcal{C}(\Gamma)$, a set $\mathcal{C}(\Gamma \vdash A)$ of elements of the family A over Γ
- comprehension structure. . .

Category with Families (CwF)

A CwF is given by... plus a comprehension structure:

[modelling the basic properties of the judgements of MLTT, independent of any particular type-forming constructs]

Category with Families (CwF)

A CwF is given by. . . plus a comprehension structure:

for each $\Gamma \in \mathcal{C}$ and $A \in \mathcal{C}(\Gamma)$, an object $\Gamma.A \in \mathcal{C}$, projection morphism $\mathrm{p}_A \in \mathcal{C}(\Gamma.A, \Gamma)$, generic element $\mathrm{q}_A \in \mathcal{C}(\Gamma.A \vdash A[\mathrm{p}])$ and pairing operation

$$\frac{\gamma \in \mathcal{C}(\Delta, \Gamma) \qquad a \in \mathcal{C}(\Delta \vdash A[\gamma])}{\langle \gamma\,, a \rangle \in \mathcal{C}(\Delta, \Gamma.A)}$$

satisfying
$$\begin{cases} \mathrm{p}_A \circ \langle \gamma\,, a \rangle &= \gamma \\ \mathrm{q}_A[\langle \gamma\,, a \rangle] &= a \\ \langle \gamma\,, a \rangle \circ \delta &= \langle \gamma \circ \delta\,, a[\delta] \rangle \\ \langle p_A\,, \mathrm{q}_A \rangle &= \mathbf{id}_{\Gamma.A} \end{cases}$$

Overview of the CCHM model

- Every topos $\mathcal{E}$ has an associated CwF so that families over $\Gamma \in \mathcal{E}$ equivalent to morphisms with cod Γ, $\mathcal{E}(\Gamma) \simeq \mathcal{E}/\Gamma$.

  [These are models of ETT, with the identification type for $A \to \Gamma$ given by the diagonal $A \xrightarrow{\Delta} A \times_{\Gamma} A$.]

Overview of the CCHM model

- Every topos $\mathcal{E}$ has an associated CwF so that families over $\Gamma \in \mathcal{E}$ equivalent to morphisms with cod Γ, $\mathcal{E}(\Gamma) \simeq \mathcal{E}/\Gamma$. For CCHM we take $\mathcal{E} \equiv \mathbf{Set}^{C^{\mathrm{op}}}$ where C is the small category of free, finitely generated De Morgan algebras (more on those later).

Overview of the CCHM model

- Every topos $\mathcal{E}$ has an associated CwF so that families over $\Gamma \in \mathcal{E}$ equivalent to morphisms with cod Γ, $\mathcal{E}(\Gamma) \simeq \mathcal{E}/\Gamma$.
- Using an interval $0, 1 : 1 \rightrightarrows \mathbb{I}$ and a subobject of cofibrant propositions $\mathbb{F} \rightarrowtail \Omega$ in the topos $\mathcal{E}$, one defines a notion of fibration structure $\alpha \in \mathbf{Fib}(A)$ on families $A \in \mathcal{E}(\Gamma)$, giving a new CwF $\mathcal{F}$ (based on $\mathcal{E}$) with $\mathcal{F}(\Gamma) \equiv \sum_{A \in \mathcal{E}(\Gamma)} \mathbf{Fib}(A)$ and $\mathcal{F}(\Gamma \vdash (A, \alpha)) \equiv \mathcal{E}(\Gamma \vdash A)$.

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- Working in the internal ETT of a topos $\mathcal{E}$, [OP] identifies axioms on $0, 1 : 1 \rightrightarrows \mathbb{I}$ and $\mathbb{F} \rightarrowtail \Omega$ that ensure we get a model of intensional MLTT:

  – fibrations are closed under $\mathcal{E}$'s Π, Σ, W, ...
  e.g. have $\mathbf{Fib}(A) \to \mathbf{Fib}(B) \to \mathbf{Fib}(\Pi\, A\, B)$

  – path objects $\mathbb{I} \to \Gamma$ yield (propositional, non-truncated) identification types in $\mathcal{F}$

Overview of the CCHM model

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- Working in the internal ETT of a topos $\mathcal{E}$, [OP] identifies axioms on $0, 1 : 1 \rightrightarrows \mathbb{I}$ and $\mathbb{F} \rightarrowtail \Omega$ that ensure we get a model of intensional MLTT.
- When $\mathcal{E} = \mathbf{Set}^{C^{op}}$ with C the category of free finitely generated De Morgan algebras, [CCHM] show that Hofmann-Streicher universe construction in $\mathcal{E}$ can be extended so that $\mathcal{F}$ is a model of UTT.

Overview of the CCHM model

- Every topos $\mathcal{E}$ has an associated CwF so that families over $\Gamma \in \mathcal{E}$ equivalent to morphisms with cod Γ, $\mathcal{E}(\Gamma) \simeq \mathcal{E}/\Gamma$.
- Using an interval $0, 1 : 1 \rightrightarrows \mathbb{I}$ and a subobject of cofibrant

The details are complicated!

Here I give an equivalent, "nominal" formulation of $\mathbf{Set}^{C^{\mathrm{op}}}$ as a topos of finitely supported $\mathbb{M}$-sets that may enable a simpler treatment.

Path types in the new formulation look like name abstraction sets from the theory of nominal sets.

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- Overview of the Cohen-Coquand-Huber-Mörtberg presheaf model of univalent type theory [CCHM,OP]
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- Paths objects
- Cofibrant propositions and fibrant families
- A univalent universe [CCHM]

CwF of $\mathbb{M}$-sets, $\mathbf{Set}^{\mathbb{M}}$

Fix a monoid $(\mathbb{M}, \_ \circ \_, \mathbf{id})$.

$$m \circ (m' \circ m'') = (m \circ m') \circ m''$$
$$\mathbf{id} \circ m = m$$
$$m \circ \mathbf{id} = m$$

(w.l.o.g. $\mathbb{M}$ is a set of endofunctions)

CwF of $\mathbb{M}$-sets, $\mathbf{Set}^{\mathbb{M}}$

Objects $\Gamma \in \mathbf{Set}^{\mathbb{M}}$ are sets equipped with an $\mathbb{M}$-action

$$m \in \mathbb{M}, x \in \Gamma \mapsto m \cdot x \in \Gamma$$

$$m' \cdot (m \cdot x) = (m' \circ m) \cdot x$$

$$\mathbf{id} \cdot x = x$$

CwF of $\mathbb{M}$-sets, $\mathbf{Set}^{\mathbb{M}}$

Objects $\Gamma \in \mathbf{Set}^{\mathbb{M}}$ are sets equipped with an $\mathbb{M}$-action

Morphisms $\gamma \in \mathbf{Set}^{\mathbb{M}}(\Delta, \Gamma)$ are functions preserving the $\mathbb{M}$-action

$$m \cdot (\gamma\, x) = \gamma(m \cdot x)$$

CwF of $\mathbb{M}$-sets, $\mathbf{Set}^{\mathbb{M}}$

Objects $\Gamma \in \mathbf{Set}^{\mathbb{M}}$ are sets equipped with an $\mathbb{M}$-action

Morphisms $\gamma \in \mathbf{Set}^{\mathbb{M}}(\Delta, \Gamma)$ are functions preserving the $\mathbb{M}$-action

Families $A \in \mathbf{Set}^{\mathbb{M}}(\Gamma)$ are families of sets $(A\, x \in \mathbf{Set} \mid x \in \Gamma)$ equipped with a dependently-typed $\mathbb{M}$-action

$$m \in \mathbb{M}, a \in A\, x \mapsto m \cdot a \in A(m \cdot x) \quad (x \in \Gamma)$$

$$m' \cdot (m \cdot a) = (m' \circ m) \cdot a$$

$$\mathbf{id} \cdot a = a$$

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Elements $\alpha \in \mathbf{Set}^{\mathbb{M}}(\Gamma \vdash A)$ are dependent functions $\alpha \in \prod_{x \in \Gamma} A\,x$ preserving the $\mathbb{M}$-action

$$m \cdot (\alpha\,x) = \alpha(m \cdot x)$$

CwF of $\mathbb{M}$-sets, $\mathbf{Set}^{\mathbb{M}}$

Comprehension structure:

$$\Gamma.A \equiv \textstyle\sum_{x \in \Gamma} A\, x$$
$$m \cdot (x, a) \equiv (m \cdot x, m \cdot a)$$
$$\mathrm{p}_A(x, a) \equiv x$$
$$\mathrm{q}_A(x, a) \equiv a$$
$$\langle \gamma\,, \alpha \rangle\, y \equiv (\gamma\, y, \alpha\, y)$$

CwF of $\mathbb{M}$-sets, $\mathbf{Set}^{\mathbb{M}}$

Σ-types [Hof, Definition 3.15]:

given $\Gamma \in \mathbf{Set}^{\mathbb{M}}$, $A \in \mathbf{Set}^{\mathbb{M}}(\Gamma)$ and $B \in \mathbf{Set}^{\mathbb{M}}(\Gamma.A)$, we get

$$\Sigma\, A\, B \in \mathbf{Set}^{\mathbb{M}}(\Gamma)$$

with

$$(\Sigma\, A\, B)\, x \equiv \textstyle\sum_{a \in A\, x} B(x, a)$$
$$m \cdot (a, b) \equiv (m \cdot a, m \cdot b)$$

etc

CwF of $\mathbb{M}$-sets, $\mathbf{Set}^{\mathbb{M}}$

Π-types [Hof, Definition 3.18]:

given $\Gamma \in \mathbf{Set}^{\mathbb{M}}$, $A \in \mathbf{Set}^{\mathbb{M}}(\Gamma)$ and $B \in \mathbf{Set}^{\mathbb{M}}(\Gamma.A)$, we get

$$\Pi\, A\, B \in \mathbf{Set}^{\mathbb{M}}(\Gamma)$$

where for each $x \in \Gamma$, $(\Pi\, A\, B)\, x$ is the set

$$\{f \in \textstyle\prod_{m \in \mathbb{M}} \prod_{a \in A(m \cdot x)} B(m \cdot x, a) \mid (\forall m, m', a)\; m' \cdot (f\, m\, a) = f(m' \circ m)(m' \cdot a)\}$$

with $\mathbb{M}$-action given by $(m' \cdot f)\, m\, a \equiv f(m \circ m')\, a$.

Etc.

Topos structure of $\mathbf{Set}^{\mathbb{M}}$

Limits (& colimits) are created by the forgetful functor $U : \mathbf{Set}^{\mathbb{M}} \to \mathbf{Set}$.

Subobjects of $\Gamma \in \mathbf{Set}^{\mathbb{M}}$ correspond to subsets of $U\,\Gamma \in \mathbf{Set}$ that are closed under the $\mathbb{M}$-action.

Subobject classifier:

$$\Omega \equiv \{\varphi \subseteq \mathbb{M} \mid (\forall m, m')\; m \in \varphi \Rightarrow m' \circ m \in \varphi\}$$
$$m \cdot \varphi \equiv \{m' \in \mathbb{M} \mid m' \circ m \in \varphi\}$$

so $m' \in m \cdot \varphi \;\Leftrightarrow\; m' \circ m \in \varphi$

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Truth $\top \in \mathbf{Set}^{\mathbb{M}}(\mathbf{1}, \Omega)$ is $\top(\mathbf{0}) \equiv \mathbb{M}$

Classifier of $S \rightarrowtail \Gamma$ is $\chi_S \in \mathbf{Set}^{\mathbb{M}}(\Gamma, \Omega)$ where

$$\chi_S\, x \equiv \{m \in \mathbb{M} \mid m \cdot x \in S\}$$

The CCHM monoid

From now on we take
$\mathbb{M}$ to be the monoid of finitary endomorphisms
of the free De Morgan algebra $\mathbb{I}$
on a countably infinite set $\mathfrak{J}$

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distributive lattice $(D, \vee, \wedge, \mathbf{0}, \mathbf{1})$ equipped with a function
$d \mapsto \mathbf{1} - d$ which is involutive $\mathbf{1} - (\mathbf{1} - d) = d$
and satisfies De Morgan's Law $\mathbf{1} - (d_1 \vee d_2) = (\mathbf{1} - d_1) \wedge (\mathbf{1} - d_2)$

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From now on we take
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of the free De Morgan algebra $\mathbb{I}$
on a countably infinite set $\mathfrak{I}$

we call elements of $\mathfrak{I}$ cartesian **directions**
and write them as $i, j, k, \ldots$

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elements of $\mathbb{I}$ are equivalence classes for the equational theory of De Morgan algebra of 'De Morgan polynomials'

$$d ::= i \mid \mathbf{0} \mid \mathbf{1} \mid d \vee d \mid d \wedge d \mid \mathbf{1} - d \qquad (i \in \mathfrak{I})$$

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elements of $\mathbb{I}$ are De Morgan algebra homomorphisms $m : \mathbb{I} \to \mathbb{I}$
for which $\mathbf{dom}(m) \equiv \{i \in \mathfrak{I} \mid m\, i \neq i\}$ is finite.

(Since $\mathbb{I}$ is the free De Morgan algebra on $\mathfrak{I}$,
m is uniquely determined as a function
by its restriction to the finite set $\mathbf{dom}(m)$.)

Notation: $(d/i) \in \mathbb{M}$ is the homomorphism m with
$\mathbf{dom}(m) = \{i\}$ and $m(i) = d$.

Finite support property

Let $\Gamma \in \mathbf{Set}^{\mathbb{M}}$ and $x \in \Gamma$

A finite set of directions $I \subseteq_{\mathbf{fin}} \mathfrak{I}$ **supports** x if
for all $m, m' \in \mathbb{M}$

$$((\forall i \in I)\; m\, i = m' i) \;\Rightarrow m \cdot x = m' \cdot x$$

(If $\mathbb{M}$ is a group (has inverses), this is equivalent to the usual nominal sets notion of finite support.)

Finite support property

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Lemma. $I \subseteq_{\mathrm{fin}} \mathfrak{I}$ supports x iff

$$(\forall i \in \mathfrak{I})\ i \notin I \ \Rightarrow\ (0/i) \cdot x = x$$

(iff $(\forall i \in \mathfrak{I})\ i \notin I \ \Rightarrow\ (1/i) \cdot x = x$)

Finite support property

Let $\mathbf{\Gamma} \in \mathbf{Set}^{\mathbb{M}}$ and $x \in \mathbf{\Gamma}$

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Lemma. $I \subseteq_{\text{fin}} \mathfrak{I}$ supports x iff

$$(\forall i \in \mathfrak{I})\ i \notin I \ \Rightarrow\ (0/i) \cdot x = x$$

The **interval**: $\mathbb{M}$ acts on $\mathbb{I}$ via function application: $m \cdot d \equiv m\, d$. With respect to this action, each $d \in \mathbb{I}$ is supported by the finite set I of directions occurring in some De Morgan polynominal representing d, since if $i \notin I$, then $(0/i)d = d$.

De Morgan sets

The category $\mathbf{Dms}$ of **De Morgan sets** is the full subcategory of $\mathbf{Set}^{\mathbb{M}}$ consisting of those $\mathbb{M}$-sets Γ such that every $x \in \Gamma$ possesses a finite support.

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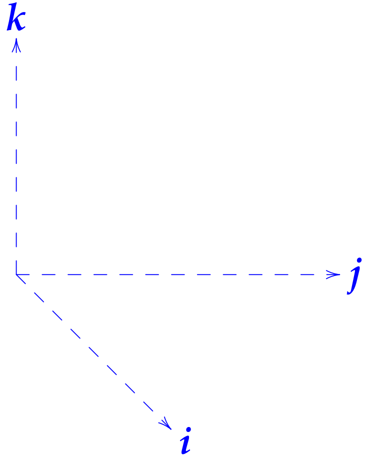
Hence $\mathbf{Dms}$ is a topos. In fact:

Theorem. (Orton, AMP) $\mathbf{Dms}$ is equivalent to the presheaf topos $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ used in [CCHM].

($\mathcal{C}^{\mathrm{op}}$ is the category of free, finitely generated De Morgan algebras and homomorphisms.)

One can view De Morgan sets $\mathbf{\Gamma} \in \mathbf{Dms}$ as sets whose elements depend implicitly on finitely many (named) dimensions $i, j, k, \ldots$



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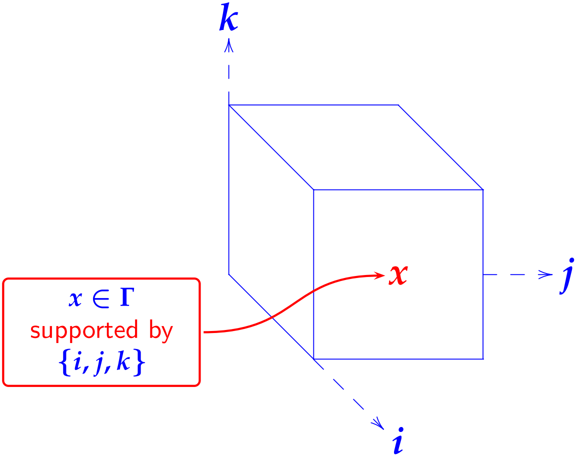

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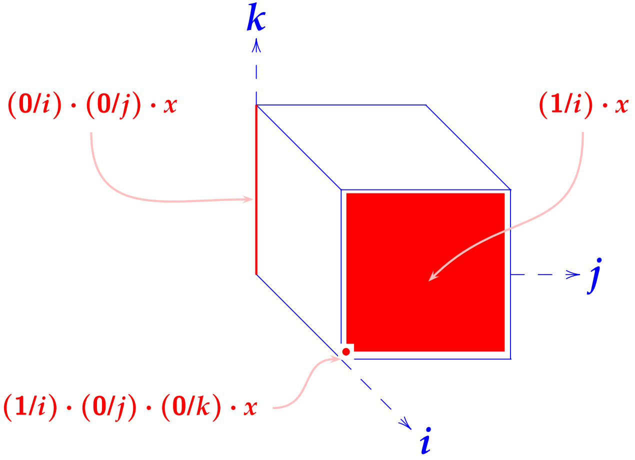

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in the [CCHM] version using $\mathbf{Set}^{C^{\mathrm{op}}}$, dependency is explicit $\rightsquigarrow$ "weakening hell"

Other toposes of interest for modelling Homotopy Type Theory can be presented (usefully?) as categories of finitely supported $\boldsymbol{M}$-sets for various monoids $\boldsymbol{M}$.

E.g. other variations on the notion of "cubical set"

Theorem. [Pit] The presheaf category on Grothendieck's "smallest test category" (non-trivial bipointed finite sets)$^{\mathbf{op}}$ is equivalent to the category of finitely supported $\boldsymbol{M}$-sets where $\boldsymbol{M}$ is the monoid of endofunctions on $\{\bot\} \cup \mathbb{Z} \cup \{\top\}$ preserving $\bot$ and $\top$.

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E.g. other variations on the notion of "cubical set" but also simplicial sets:

Theorem. (Faber) The presheaf topos $\mathbf{Set}^{\Delta^{\mathrm{op}}}$ of simplicial sets is equivalent to the category of finitely supported $\boldsymbol{M}$-sets where $\boldsymbol{M}$ is the monoid of order-preserving endofunctions on

$$\{\bot \leq \cdots - \mathbf{2} \leq -\mathbf{1} \leq \mathbf{0} \leq \mathbf{1} \leq \mathbf{2} \leq \cdots \leq \top\}$$

preserving $\bot$ and $\top$.