Axiomatizing Cubical Sets Models of Univalent Foundations

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HoTT/UF Workshop 2018

HoTT/UF from the outside in

Why study models of univalent type theory?
(instead of just developing univalent foundations)

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  as a concept, as opposed to a particular formal axiom, and its relation to other foundational concepts & axioms

- higher inductive types

  formalization, properties

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This talk concentrates on the first point, but the second one is probably of more importance in the long term (cf. CoC vs CIC).

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Wanted:

- simpler proofs of univalence for existing models
- new models
- [better understanding of HITs in models]

HoTT/UF from the outside in

Why study models of univalent type theory?
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Some possible approaches:

- Direct calculations in set/type theory with presheaves (or nominal variations thereof) [wood from the trees]
- Categorical algebra (theory of model categories) [strictness issues]

HoTT/UF from the outside in

Why study models of univalent type theory?
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Some possible approaches:

- Direct calculations in set/type theory with presheaves (or nominal variations thereof).
- Categorical algebra (theory of model categories).
- **Categorical logic**
  Here we describe how, in a version of type theory interpretable in any elementary topos with countably many universes $\Omega : \mathcal{S}_0 : \mathcal{S}_1 : \mathcal{S}_2 : \cdots$, there are
  axioms for $\begin{cases} \text{interval object } 0, 1 : \mathbf{1} \rightrightarrows \mathbb{I} \\ \text{cofibrant propositions } \mathbf{Cof} \rightarrowtail \Omega \end{cases}$
  that suffice for a version of the model of univalence of **Coquand *et al*.**

Topos theory background

Elementary topos $\mathcal{E}$ = cartesian closed category with subobject classifier Ω (& natural number object)

Toposes are the category-theoretic version of theories in extensional impredicative higher-order intuitionistic predicate calculus.

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Can make a category-with-families (CwF) out of $\mathcal{E}$ and soundly interpret Extensional Martin-Löf Type Theory (EMLTT) in it

Type Theory		CwF $\mathcal{E}$
context	Γ	object Γ
type (of size n) in context	$\Gamma \vdash_n A$	morphism $\Gamma \xrightarrow{A} \mathcal{S}_n$
typed term in context	$\Gamma \vdash a : A$	section $\begin{array}{ccc} & & \tilde{\mathcal{S}}_n \\ & \nearrow^{a} & \downarrow \\ \Gamma & \xrightarrow[A]{} & \mathcal{S}_n \end{array}$
judgemental equality	$\Gamma \vdash a = a' : A$	equality of morphisms
extensional identity types		cartesian diagonals

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For the moment, I work in a meta-theory in which the category $\mathbf{Set}$ is an elementary topos with universes.
(ZFC or IZF, not CZF, + Grothendieck universes)

Given a category $\mathbf{C}$ in $\mathbf{Set}$ we get a topos $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ of $\mathbf{Set}$-valued presheaves.

CCHM Univalent Universe

C. Cohen, T. Coquand, S. Huber and A. Mörtberg, *Cubical type theory: a constructive interpretation of the univalence axiom* [arXiv:1611.02108]

Uses categories-with-families (CwF) semantics of type theory for the CwF associated with presheaf topos

$$\mathcal{E} = \mathbf{Set}^{\square^{\mathrm{op}}}$$

where $\square$ is the Lawvere theory of De Morgan algebras.

Axiomatic CCHM

Starting with any topos $\mathcal{E}$ satisfying some

axioms for $\begin{cases} \text{interval object } 0, 1 : \mathbf{1} \rightrightarrows \mathbb{I} \\ \text{cofibrant propositions } \mathbf{Cof} \rightarrowtail \Omega \end{cases}$

one gets a model of MLTT + univalence

by building a new CwF $\mathcal{F}$ out of $\mathcal{E}$:

- objects of $\mathcal{F}$ are the objects of $\mathcal{E}$
- families in $\mathcal{F}$: $\mathcal{F}_n(\Gamma) \triangleq \sum_{A:\Gamma\to\mathcal{S}_n} \mathbf{Fib}_n\, A$ where
  $\mathbf{Fib}_n\, A$ = set of CCHM fibration structures on $A : \Gamma \to \mathcal{S}_n$
- elements of $(A, \alpha) \in \mathcal{F}_n(\Gamma)$ are elements of A in $\mathcal{E}$

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. . . is a form of (uniform) Kan-filling operation w.r.t. cofibrant propositions:

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Given a family of types $A : \Gamma \to \mathcal{S}_n$ (for some fixed n),
a CCHM fibration structure $\alpha : \mathbf{Fib}_n\, A$ maps

path in Γ	$p : \mathbb{I} \to \Gamma$
cofibrant partial path over p	$f : \prod_{i:\mathbb{I}}(\varphi \to A(p\, i))$ with $\varphi : \mathbf{Cof}$
extension of f at $\mathtt{0}$	$a_0 : A(p\, \mathtt{0})$ with $f\, \mathtt{0} \nearrow a_0$

to

extension of f at $\mathtt{1}$	$a_1 : A(p\, \mathtt{1})$ with $f\, \mathtt{1} \nearrow a_1$

where extension relation for $\varphi : \mathbf{Cof}$, $f : \varphi \to \Gamma$ and $x : \Gamma$ is

$$f \nearrow x \triangleq \prod_{u:\varphi}(f\, u = x) \quad \text{“}f \text{ agrees with } x \text{ where defined”}$$

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extension of f at 0	$a_0 : A(p\, 0)$ with $f\, 0 \nearrow a_0$

to

extension of f at 1	$a_1 : A(p\, 1)$ with $f\, 1 \nearrow a_1$

Some simple properties of $\mathbb{I}$ and $\mathbf{Cof}$ enable one to prove that the existence of fibration structure is preserved under forming Σ-types, Π-types, (propositional) identity types,...

What about universes of fibrations? We get them via “tinyness” of the interval...

Tiny interval

$\mathbb{I} \in \mathcal{E}$ is tiny if $(\_)^{\mathbb{I}}$ has a right adjoint $\sqrt{}(\_)$

$$\frac{\Gamma^{\mathbb{I}} \to \Delta}{\Gamma \to \sqrt{}\Delta} \quad \text{(natural bijection)}$$

preserving universe levels: $\Delta : \mathcal{S}_n \Rightarrow \sqrt{}\Delta : \mathcal{S}_n$

(notion goes back to Lawvere's work in synthetic differential geometry)

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When $\mathcal{E} = \mathbf{Set}^{\square^{\mathrm{op}}}$, the topos of cubical sets, the category $\square$ has finite products and the interval in $\mathcal{E}$ is representable: $\mathbb{I} = \square(\_, I)$.

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Hence the path functor $(\_)^{\mathbb{I}} : \mathbf{Set}^{\square^{\mathrm{op}}} \to \mathbf{Set}^{\square^{\mathrm{op}}}$ is $(\_ \times I)^*$

and so $(\_)^{\mathbb{I}}$ not only has a left adjoint $(\_ \times \mathbb{I})$, but also a right adjoint, given by right Kan extension (and hence preserving universe levels).

Tiny interval

Recall $\mathcal{F}_n(\Gamma) \triangleq \sum_{A : \Gamma \to \mathcal{S}_n} \mathbf{Fib}_n\, A$ = set of CCHM fibrations over an object $\Gamma \in \mathcal{E}$. This is functorial in Γ.

Theorem. If interval $\mathbb{I}$ is tiny, then $\mathcal{F}_n(\_) : \mathcal{E}^{\mathrm{op}} \to \mathbf{Set}$ is representable:

$\mathcal{U}_n$	(E, ν)	$\in \mathcal{F}_n(\mathcal{U}_n)$
object	generic fibration	

Theorem generalizes unpublished work of **Coquand & Sattler** for the case $\mathcal{E}$ is a presheaf topos. For proof see:

Licata-Orton-AMP-Spitters FSCD 2018 [arXiv:1801.07664]

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$$\begin{array}{ccccc} \Gamma & \qquad & (A,\alpha) & \in & \mathcal{F}_n(\Gamma) \\ \exists!\ \big\downarrow \ulcorner A,\alpha \urcorner & & \big\uparrow & & \\ \mathcal{U}_n & & (\mathsf{E},\upsilon) & \in & \mathcal{F}_n(\mathcal{U}_n) \end{array}$$

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Tiny interval

Theorem. The universes $(\mathcal{U}_n, \mathsf{E})$ of CCHM fibrations are closed under Π-types, propositional identity types and inductive types (e.g. Σ) if $\mathbb{I}$ has a weak form of binary minimum ("connection" structure) and Cof satisfies

$$\mathsf{false} \in \mathsf{Cof}$$
$$(\forall i, \varphi)\ \varphi \in \mathsf{Cof} \ \Rightarrow\ \varphi \vee i = 0 \in \mathsf{Cof}$$
$$(\forall i, \varphi)\ \varphi \in \mathsf{Cof} \ \Rightarrow\ \varphi \vee i = 1 \in \mathsf{Cof}$$

What about univalence of $(\mathcal{U}_n, \mathsf{E})$?

Univalence

Theorem. For any topos $\mathcal{E}$ with tiny $\mathbb{I}$ & $\mathbf{Cof}$ satisfying assumptions so far, there is a term of type

$$\prod_{u:\mathcal{U}_n} \mathbf{isContr}\left(\sum_{v:\mathcal{U}_n}(\mathbf{E}u \simeq \mathbf{E}v)\right)$$

if $\mathbf{Cof}$ is closed under $\forall i : \mathbb{I}$ and satisfies the isomorphism extension axiom:

$$\mathbf{iea} : \prod_{A:\mathcal{S}_n} \mathbf{Ext}\left(\sum_{B:\mathcal{S}_n}(A \cong B)\right)$$

In this case $\mathcal{U}_n$ is a fibration (over $\mathbf{1}$) and $(\mathcal{U}_n, \mathbf{E})$ is univalent.

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equivalent to the usual univalence axiom
(given suitable properties of $\mathcal{U}$)

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$$\begin{aligned}
\mathsf{isContr}\,A &\triangleq \sum_{x:A} \prod_{x':A} (x \sim x') \\
x \sim x' &\triangleq \sum_{p\,:\,\mathbb{I}\to A} (p\,0 \equiv x \wedge p\,1 \equiv x') \\
\mathsf{Ext}\,A &\triangleq \prod_{\varphi\,:\,\mathsf{Cof}} \prod_{f\,:\,\varphi\to A} \sum_{x:A} (f \nearrow x) \\
A \cong B &\triangleq \sum_{f:A\to B} \sum_{g:B\to A} (g \circ f \equiv \mathsf{id} \wedge f \circ g \equiv \mathsf{id}) \\
A \simeq B &\triangleq \sum_{f:A\to B} \prod_{y:B} \mathsf{isContr}\left(\sum_{x:A} (f\,x \sim y)\right)
\end{aligned}$$

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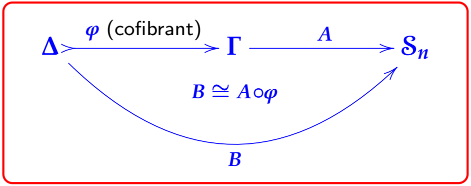

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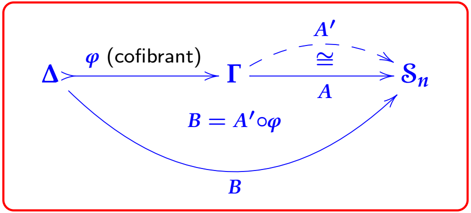

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In a presheaf topos $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$, **Cof** has an **iea** if for each $X \in \mathbf{C}$ and $S \in \mathbf{Cof}(X) \subseteq \Omega(X)$, the sieve S is a decidable subset of $\mathbf{C}/X$.
(So with classical meta-theory, always have **iea** for presheaf toposes.)

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In this case $\mathcal{U}_n$ is a fibration (over $\mathbf{1}$) and $(\mathcal{U}_n, \mathbf{E})$ is univalent.

Proof is non-trivial! It combines results from:

Cohen-Coquand-Huber-Mörtberg TYPES 2015 [arXiv:1611.02108]

Orton-AMP CSL 2016 [arXiv:1712.04864]

Licata-Orton-AMP-Spitters FSCD 2018 [arXiv:1801.07664]

Summary of axioms

- Elementary topos $\mathcal{E}$ with universes $\Omega : \mathcal{S}_0 : \mathcal{S}_1 : \mathcal{S}_2 : \cdots$
- "Interval" object $\mathbb{I}$ (in $\mathcal{S}_0$) which has distinct end-points & connection operation (& for convenience, a reversal operation) and which is tiny.
- Universe of "cofibrant" propositions **Cof** $\rightarrowtail \Omega$ containing $i \equiv 0$ and $i \equiv 1$, is closed under $\_ \vee \_$ and $\forall(i : \mathbb{I})\_$, and satisfies the isomorphism extension axiom.

Then CCHM fibrations in $\mathcal{E}$ give a model of MLTT with univalent universes w.r.t. propositional identity types given by $\mathbb{I}$-paths.

(**Swan**: can have true, judgemental identity types if **Cof** is also a dominance.)

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Next: can remove the use of impredicativity ($\boldsymbol{\Omega}$) and formalize within MLTT **plus. . .**

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Problem! Tinyness cannot be axiomatized in MLTT, because it's a global property of morphisms of $\mathcal{E}$, not an internal property of functions – there is an internal right adjoint to $(\_)^{\mathbb{I}}$ only when $\mathbb{I} \cong \mathbf{1}$.

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Next: can remove the use of impredicativity ($\boldsymbol{\Omega}$) and formalize within MLTT **plus. . .**

Tinyness: natural bijection between hom sets $\mathcal{E}(\Gamma^{\mathbb{I}}, \Delta)$ and $\mathcal{E}(\Gamma, \surd\Delta)$.

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If had natural iso of function types $(\Gamma^{\mathbb{I}} \to \Delta) \cong (\Gamma \to \surd\Delta)$

then

$$\surd\Delta \cong (1 \to \surd\Delta) \cong (1^{\mathbb{I}} \to \Delta) \cong (1 \to \Delta) \cong \Delta$$

naturally in Δ

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naturally in Δ

so $\surd \cong \mathbf{id}$

so (taking left adjoints) $(\_)^{\mathbb{I}} \cong \mathbf{id}$ ($\cong (\_)^{1}$)

so $1 \cong \mathbb{I}$

HoTT/UF from the outside in

"Crisp" Type Theory =
intensional Martin-Löf Type Theory with universes
(expressed with Agda's concrete syntax)
+ uniqueness of identity proofs
+ Hofmann-style quotient types
(⇒ function extensionality & disjunction for mere propositions)
extended with a **modality** for expressing global/local distinctions.

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ories).

- **Categorical logic**
  Here we describe how, in a version of type theory interpretable in any elementary topos with countably many universes $\Omega : \mathcal{S}_0 : \mathcal{S}_1 : \mathcal{S}_2 : \cdots$, there are
  axioms for $\begin{cases} \text{interval object } 0, 1 : 1 \rightrightarrows \mathbb{I} \\ \text{cofibrant propositions } \mathbf{Cof} \rightarrowtail \Omega \end{cases}$
  that suffice for a version of the model of univalence of **Coquand *et al***.

Crisp Type Theory

Licata-Orton-AMP-Spitters FSCD 2018 [arXiv:1801.07664]

Sources:

- Pfenning+Davis's judgemental reconstruction of modal logic [MSCS 2001]
- de Paiva+Ritter, *Fibrational modal type theory* [ENTCS 2016]
- Shulman's *spatial type theory* for real cohesive HoTT [MSCS 2017]

Crisp Type Theory

Dual context judgements:

$$\Delta \,|\, \Gamma \vdash a : A$$

crisp/global/external variables $x :: A$

cohesive/local/internal variables $x : A$

types in the crisp context Δ and terms substituted for crisp variables $x :: A$ depend only on crisp variables

Crisp Type Theory

Dual context judgements:

$$\Delta | \Gamma \vdash a : A$$

Interpretation in the CwF associated with $\mathcal{E} = \mathbf{Set}^{\square^{\mathrm{op}}}$:

$\Delta \in \mathcal{E}$, $\Gamma \in \mathcal{E}(\flat\Delta)$, $A \in \mathcal{E}(\sum(\flat\Delta)\Gamma)$, $a \in \mathcal{E}(\sum(\flat\Delta)\Gamma \vdash A)$,

where $\flat : \mathcal{E} \longrightarrow \mathcal{E}$ is the limit-preserving idempotent comonad

$\flat A =$ the constant presheaf on the set of global sections of A.

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This just follows from the fact that $\Box$ is a connected category (since it has a terminal object)

Crisp Type Theory

Dual context judgements:

$$\Delta | \Gamma \vdash a : A$$

Some of the rules:

$$\frac{}{\Delta, x :: A, \Delta' | \Gamma \vdash x : A}$$

$$\frac{\Delta | \ \vdash a : A \qquad \Delta, x :: A, \Delta' | \Gamma \vdash b : B}{\Delta, \Delta'[a/x] | \Gamma[a/x] \vdash b[a/x] : B[a/x]}$$

$$\frac{\Delta | \ \vdash A : \mathcal{S}_m \qquad \Delta, x :: A | \Gamma \vdash B : \mathcal{S}_n}{\Delta | \Gamma \vdash (x :: A) \to B : \mathcal{S}_{m \vee n}} \qquad \frac{\Delta, x :: A | \Gamma \vdash b : B}{\Delta | \Gamma \vdash \lambda(x :: A), b : (x :: A) \to B}$$

$$\frac{\Delta | \Gamma \vdash f : (x :: A) \to B \qquad \Delta | \ \vdash a : A}{\Delta | \Gamma \vdash f\, a : B[a/x]}$$

Experimental implementation: Vezzosi's Agda-flat

Axioms for tinyness in Agda-flat

$\surd : (A :: \mathcal{S}_n) \to \mathcal{S}_n$

$\mathtt{R} : \{A, B :: \mathcal{S}_n\}(f :: \wp A \to B) \to A \to \surd B$

$\mathtt{L} : \{A, B :: \mathcal{S}_n\}(g :: A \to \surd B) \to \wp A \to B$

$\mathtt{LR} : \{A, B :: \mathcal{S}_n\}\{f :: \wp A \to B\} \to \mathtt{L}(\mathtt{R}\, f) \equiv f$

$\mathtt{RL} : \{A, B :: \mathcal{S}_n\}\{g :: A \to \surd B\} \to \mathtt{R}(\mathtt{L}\, g) \equiv g$

$\mathtt{R}\wp : \{A, B, C :: \mathcal{S}_n\}(g :: A \to B)(f :: \wp B \to C) \to$
$\mathtt{R}(f \circ \wp g) \equiv \mathtt{R} f \circ g$

where $\wp(\_) \triangleq \mathbb{I} \to (\_)$.

For more, see doi.org/10.17863/CAM.22369

Conclusion

- Topos models of univalence where path types are *cartesian exponentials* make life easier compared with simplicial sets, because the path functor is fibered over $\mathcal{E}$ and we can use internal language to describe many of the constructions on the way to a univalent universe...

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  . . . but not all of them: tinyness does not internalize! (so neither does our universe construction)

  Licata-Orton-AMP-Spitters use a modal type theory ("crisp" type theory) in order to express the whole construction with a type-theoretic language.

  The whole area of Modal Type Theory is currently very active.

Conclusion

- Topos models of univalence where path types are *cartesian exponentials* make life easier compared with simplicial sets.
- The axiomatic approach helps one see the wood from the trees in existing models and to find new ones
  (see talk by Taichi Uemura in this workshop)

Conclusion

- Topos models of univalence where path types are *cartesian exponentials* make life easier compared with simplicial sets.
- The axiomatic approach helps one see the wood from the trees in existing models and to find new ones
- Nevertheless, some of the theorems on the way to univalence/fibrancy are delicate and hard work!

  We find the use of an interactive theorem proving system (Agda-flat) invaluable for developing and checking the proof – e.g. see [doi.org/10.17863/CAM.21675]

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  Are there simpler models of univalence? (must be non-truncated to qualify for our attention)

  E.g. can one avoid Kan-filling in favour of a (weak) notion of path composition?

  Why only presheaf toposes?

Conclusion

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- Nevertheless, some of the theorems on the way to univalence/fibrancy are delicate and hard work!
- Further reading:

  I. Orton and A. M. Pitts, *Axioms for Modelling Cubical Type Theory in a Topos* [arXiv:1712.04864]

  D. R. Licata, I. Orton, A. M. Pitts and B. Spitters, *Internal Universes in Models of Homotopy Type Theory* [arXiv: 1801.07664]

Thank you for your attention!