THE THEORY OF TRIPOSES

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by

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To my wife, Susan.

PREFACE

This dissertation is not substantially the same as any that I have submitted for a degree or diploma or other qualification at any other University. Except where noted in the Introduction, the results presented herein are the product of my own original research.

I would like to express my very great thanks firstly to my supervisor Dr.Peter Johnstone and also to Dr.Martin Hyland, for all the help and encouragement they have given me. I would also like to acknowledge the financial support of the Science Research Council, and Trinity and St.John's Colleges, Cambridge.

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CONTENTS

0. Introduction	1
1. Definition and examples	8
2. The basic construction	20
3. The "constant objects" functor	31
4. Morphisms over a fixed base	46
5. Topologies and sub-triposes	58
6. Iteration	67
7. The effective topos construction	82
References	106

0. INTRODUCTION

I want to begin by calling to mind the theory HA^{ω} of <u>Heyting Arithmetic of all Finite Types</u> (see [18], for example), where we consider types built up from a base type 0 (for N) using $\times$ and $\rightarrow$: thus if α,β are types, so are $\alpha\times\beta$ (product type) and $\alpha\rightarrow\beta$ (exponential type). If E is a topos with natural number object N, the cartesian closed structure of E allows us to build a model of HA^{ω} in a very natural way with $N_0 = N$, $N_{\alpha\times\beta} = N_\alpha \times N_\beta$ and $N_{\alpha\rightarrow\beta} = (N_\beta)^{N_\alpha}$: call these the <u>full</u> finite types over N in E.

A somewhat different approach to modelling HA^{ω} is to give an interpretation of it within HA. This is the purpose of the <u>hereditarily extensional effective operations</u>, which we can define as follows:

Subsets $E_\alpha \subseteq N$ and equivalence relations $=_\alpha$ on E_α are built up inductively on the type symbols α according to the clauses

(i) $E_0 = \mathbb{N}$;

(ii) $n =_0 n'$ iff $n = n'$;

(iii) $E_{\alpha\times\beta} = \{<n,m> \mid n \in E_\alpha \text{ and } m \in E_\beta\}$;

(iv) $n =_{\alpha\times\beta} n'$ iff $(n)_0 =_\alpha (n')_0$ and $(n)_1 =_\beta (n')_1$;

(v) $E_{\alpha\rightarrow\beta} = \{n \mid \forall m,m' \in E_\alpha (m =_\alpha m' \Rightarrow \exists k,k' \in E_\beta (\{n\}(m) = k, \{n\}(m') = k' \text{ and } k =_\beta k'))\}$

(vi) $n =_{\alpha\rightarrow\beta} n'$ iff for all $m \in E_\alpha$ $\{n\}(m) =_\beta \{n'\}(m)$.

(Here $< \cdot , \cdot >$ and $(\cdot)_0, (\cdot)_1$ are primitive recursive pairing and unpairing functions, and $\{n\}(m)$ is the value at m (if defined) of the partial recursive function with index n.)

Then we define the extensional effective operations of type α to comprise the quotient set $E_\alpha/=_\alpha$.

Now J.M.E.Hyland has shown that we can regard these extensional effective operations as the full finite types over the natural number object in a particular (elementary) topos, which has come to be called the <u>effective</u> <u>topos</u> and denoted Eff. Let me briefly indicate how this comes about.

Firstly recast the definition of the effective operations by defining maps $I_\alpha : \mathbb{N}_\alpha \longrightarrow P\mathbb{N}$ inductively on the types α as follows:

(a) $\mathbb{N}_0 = \mathbb{N}$ and $I_0(n) = \{n\}$;

(b) $\mathbb{N}_{\alpha\times\beta} = \mathbb{N}_\alpha \times \mathbb{N}_\beta$ and $I_{\alpha\times\beta}(x,y) = \{\langle n,m\rangle \mid n\in I_\alpha(x) \text{ and } m\in I_\beta(y)\}$;

(c) $\mathbb{N}_{\alpha\to\beta} = \{f\in(\mathbb{N}_\beta)^{\mathbb{N}_\alpha} \mid I_{\alpha\to\beta}(f) \text{ is inhabited}\}$, where

$$I_{\alpha\to\beta}(f) = \{n \mid \forall x\in\mathbb{N}_\alpha\ \forall m\in I_\alpha(x)\ \exists k\in I_\beta(fx)\ \{n\}(m)=k\}.$$

Then $E_\alpha = \bigcup\{I_\alpha(x) \mid x\in\mathbb{N}_\alpha\}$, and the extensional effective operations of type α, $E_\alpha/{=_\alpha}$, are in bijection with the $\mathbb{N}_\alpha$. Let us think of $I_\alpha(x)$ as measuring the extent to which $x\in\mathbb{N}_\alpha$ is in the type α; in fact let us write $[\![x\in N_\alpha]\!]$ for the set $I_\alpha(x)$ $(x\in\mathbb{N}_\alpha)$. So we are regarding subsets of $\mathbb{N}$ as truth-values (in the same way that we regard open subsets of a space X or elements of a locale H as truth-values). The clauses (b) and (c) suggest how to form conjunctions and implications of these new truth-values: given $p,q\subseteq\mathbb{N}$ let

$$p\wedge q = \{<n,m> \mid n\in p \text{ and } m\in q\}$$

and

$$p\to q = \{n \mid \forall m\in p\ \exists k\in q\ \{n\}(m)=k\}.$$

Then (b) becomes

$$[\![(x,y)\in N_{\alpha\times\beta}]\!] = [\![x\in N_\alpha]\!] \wedge [\![y\in N_\beta]\!],$$

and (c) becomes

$$[\![f\in N_{\alpha\to\beta}]\!] = \bigcap_{x\in\mathbb{N}_\alpha} [\![x\in N_\alpha]\!] \to [\![fx\in N_\beta]\!].$$

The latter suggests that universal quantification is given by intersection, so that the right-hand side could be written

$$[\![\forall x(x\in N_\alpha \to fx\in N_\beta)]\!].$$

This looks promising. Note in particular that whilst still resembling H-valued logic, the above constructions mimick the notion of realizability (c.f. [18]). Now for the latter one is interested in when there exists a number realizing a particular formula: correspondingly it seems that we should regard a formula as true in this semantics if the subset of $\mathbb{N}$ that is its interpretation contains some number (just as $x \in \mathbb{N}_\alpha$ was an effective operation as $[\![x \in N_\alpha]\!] = I_\alpha(x)$ was inhabited). The effect of this definition of truth on the truth-values $p \subseteq \mathbb{N}$ is to collapse them to $2 = \{\bot, \top\}$, since $p \to q$ is inhabited iff q is or p is not. So does the structure we are building also collapse to the classical 2-valued semantics? Of course the answer is no, and we have already seen why not. For a formula such as

$$\forall x(x \in N_\alpha \to fx \in N_\beta)$$

becomes true in the semantics only if there is some $n \in \mathbb{N}$ which is in all the $[\![x \in N_\alpha]\!] \to [\![fx \in N_\beta]\!]$ $(x \in \mathbb{N}_\alpha)$ simultaneously.

To build the effective topos we just mimick the construction of H-valued sets (see [6] for example). Thus as objects take sets X together with an equality relation $x,x' \longmapsto [\![x = x']\!] \subseteq \mathbb{N}$ on $X \times X$ which should be symmetric and transitive, i.e.

$$\bigcap_{x,x'} [\![x = x']\!] \to [\![x' = x]\!]$$

and

$$\bigcap_{x,x',x''} [\![x = x']\!] \wedge [\![x' = x'']\!] \to [\![x = x'']\!]$$

are to be inhabited. As maps $(X,=) \longrightarrow (Y,=)$ take (equivalence classes of) strict functional relations $X \times Y \longrightarrow P\mathbb{N}$. Hence $\mathbb{N}_\alpha$ with equality $x,x' \longmapsto I(x) \cap I(x')$ becomes an object N_α of Eff. Eff is a topos (but not a Grothendieck one) and within that topos the N_α are (isomorphic to) the full finite types defined at the beginning of this introduction. (I should mention that independently of Hyland, W.Powell formulated the above semantics and used it to build a hierarchy that stands in the same relation to the effective topos as the hierarchy V^H does to the topos of sheaves over the complete Heyting algebra (locale) H; see [15].)

Originally the effective topos was given, much as I have described it, as a topos of Ω-valued sets where Ω (=$P\mathbb{N}$ with the realizability structures defined above) was a "model for second order propositional logic", directly generalising the construction of the topos of H-valued sets from a locale H. It was while studying this construction that I formulated the notion of <u>tripos</u>. (The name was coined by P.T. Johnstone and is an acronym, standing for <u>Topos</u>-<u>Representing</u> <u>Indexed</u> <u>Pre-Ordered</u> <u>Set</u>.) This kind of structure is both a generalisation and a simplification of the original models for second order propositional logic: there are more examples of it and the definition allows the use of techniques from categorical logic and from indexed category theory to advantage. (I shall draw upon the language of the latter theory in what follows: see [14] for an account of it.) As a way of codifying logic, a tripos is a particular instance of F. W. Lawvere's "hyperdoctrines" [12] and is closely related to the structures in [3]. What distinguishes it from the former is the requirement in the definition of tripos of a generic predicate, whilst it differs from the latter by not mentioning (extensional) equality. As a result there is a happy balance: the definition is lax enough to allow examples as dissimilar as locales and realizability structures, whilst still ensuring that a rather rich theory emerges in the chapters that follow. Let me outline how this theory is developed in what follows.

Chapter 1 begins with the definition of tripos and various definability results are given which reduce the amount of structure needed to specify a tripos. The various known examples are then given. Of these, two are in fact broad classes of related structures: complete Heyting algebras give <u>localic</u> triposes and partial combinatory algebras give <u>realizability</u> triposes. An interesting non-example is also given: it shows that even without a generic predicate, an indexed pre-order may still represent a topos - in this case a Boolean topos with any specified (possibly incomplete) Boolean algebra of truth-values (c.f. 1.8 and 2.9). On the other hand, the power-full results of Chapters 3-6 will not apply to such toposes.

In Chapter 2, starting from a C-tripos **P**, the topos it represents, C[**P**] is constructed. The method is a direct generalisation of that used in constructing H-valued sets from H. If we think of a tripos as giving a semantics for higher-order Intuitionistic logic <u>without</u> equality, then this passage to C[**P**] is one of adding equality plus the Axiom of Extensionality at all types. Conversely, given that a topos is presented as C[**P**], its internal logic is described in terms of the "external" logic of **P**.

In Chapter 3 the constant-objects functor $\Delta_{\mathbf{P}}: C \longrightarrow C[\mathbf{P}]$ is defined, and those functors $C \longrightarrow E$ from C into a topos which arise in this way are characterised: in general $\Delta_{\mathbf{P}}$ does <u>not</u> have a right adjoint (global sections) and the characterisation is not as straightfoward as for the particular case of a localic extension. The chapter ends with the useful result (3.14) that existential and universal quantification in a tripos **P** can be given in a simple, generic way iff the functor $\Delta_{\mathbf{P}}$ is regular.

Chapter 4 examines the connection between (indexed) functors between triposes and functors between the toposes they represent. In particular the notion of a geometric morphism between triposes is developed and a pleasing correspondence is exhibited between such morphisms $\mathsf{P} \longrightarrow \mathsf{R}$ and geometric morphisms $C[\mathsf{P}] \longrightarrow C[\mathsf{R}]$ between the toposes whose inverse image functors preserve constant objects. In demonstrating this, essential use is made of membership predicates (or a generic predicate) given in the definition of tripos: they allow a weak form of "sheafification" to be carried out, which is nevertheless sufficient for constructing direct image functors.

Chapter 5 continues the development of Chapter 4 by examining inclusions and sheaf sub-triposes. The results are employed in investigating the sheaf sub-toposes of the effective topos.

Chapter 6 presents an extremely useful result on iterating the construction of Chapter 2. Using it, localic and realizability triposes can be combined in various orders to obtain new triposes, and examples of this are given. In particular it is shown that if the effective topos construction is applied to a realizability topos, another realizability topos is obtained.

The final Chapter 7 applies the results obtained so far to studying the construction which sends a topos E (with natural number object) to the effective topos eE defined from it (<u>the</u> effective topos described at the beginning of this introduction being eSet). There are two principal results. Firstly the assignment $E \longmapsto eE$ is the object part of a functor and the topos eE (or more precisely the functor $\Delta : E \longrightarrow eE$ defined in Chapter 3) has a certain "universal property" (Theorem 7.11); this gives a categorical characterisation of the effective topos construction. Secondly, the extent to which the assignment $E \longmapsto eE$ fails to be idempotent is measured by a monad (Theorem 7.19). The category of algebras for this monad is identified for the special case of realizability toposes. It is here that the consequences of the "geometrically" oriented theory of the preceding chapters are worked out for the "recursion-theoretic" realizability toposes.

1. DEFINITION AND EXAMPLES

Let C be a category with finite limits (in fact, just finite products will do).

<u>1.1 Definition</u>

A <u>C-tripos</u>, P, is a C-indexed category together with the following structure:

(a) For each object I of C, the category PI is actually a pre-ordered set (that is, there is at most one arrow between any two objects); write $\vdash_I$ for the pre-order on PI.

In addition, the category PI is to have finite limits and colimits, and be cartesian closed (and we are given specific binary operations of meet ($\wedge_I$), join ($\vee_I$) and Heyting implication ($\rightarrow_I$), together with distinguished top ($\top_I$) and bottom ($\bot_I$) elements). We shall call pre-ordered sets with this structure <u>Heyting pre-algebras</u>.

(b) P is complete and cocomplete as a C-indexed category. That is, for each map $f: I \longrightarrow J$ in C we are given left and right adjoints to the functor Pf, denoted $\exists f$, $\forall f$ respectively, which satisfy the Beck condition:

given a pullback square $\begin{array}{ccc} P & \xrightarrow{h} & J \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle g} \\ I & \xrightarrow{f} & K \end{array}$ in C, we have

$Pg \cdot \forall f \dashv\vdash_J \forall h \cdot Pk$ and $Pg \cdot \exists f \dashv\vdash_J \exists h \cdot Pk$.

(Actually, if we have the Beck condition for $\forall$, we have it for $\exists$ as well by taking left adjoints.)

Furthermore we require that each Pf preserve the implication $\rightarrow_J$ (it already preserves meets and joins by virtue of the above).

(c) For each object I of C there is PI in C and $\in_I$ in $P(I\times PI)$, such that given any φ in $P(I\times J)$ there is a map $\{\varphi\}: J \longrightarrow PI$ in C with $P(id_I \times \{\varphi\}) \in_I \dashv\vdash \varphi$ in $P(I\times J)$. (We will call $\in_I$ the <u>membership predicate</u> for I.)

A tripos is a structure in which to interpret logic. Specifically, one should think of the category C as carrying the "type" and "term" structure, and of each PI as a collection of (non-standard) "predicates" on I, pre-ordered by "entailment", $\vdash_I$. Part (a) of the definition deals with the Intuitionistic Propositional Calculus, part (b) with substitution and quantification, and part (c) with higher-order logic. The only thing missing is equality.

1.2 Definability results

Just as some parts of logic can be defined in terms of others, so some of the structure required by Definition 1.1 can be given in terms of the rest. Specifically $\top, \wedge, \bot, \vee$ and $\exists$ can always be defined in terms of $\rightarrow$, $\forall$ and $\in$:

$$\top = \forall\pi(\in_I \rightarrow \in_I),$$

$$\varphi \wedge \varphi' = \forall\pi((P\pi\varphi \rightarrow (P\pi\varphi' \rightarrow \in_I)) \rightarrow \in_I),$$

$$\bot = \forall\pi\in_I,$$

$$\varphi \vee \varphi' = \forall\pi((P\pi\varphi \rightarrow \in_I \wedge P\pi\varphi' \rightarrow \in_I) \rightarrow \in_I),$$

and $\exists f(\psi) = \forall\pi'(\forall(f \times id)(P\pi\psi \rightarrow P(f \times id)\in_I \rightarrow \in_I)$,

where $\pi: I \times PI \longrightarrow I$, $\pi': J \times PJ \longrightarrow J$ are the projections and $f: J \longrightarrow I$. See 1.4 of [9] for the details.

In Definition 1.1 we assumed as little about C as possible. However, as indicated in the introduction we intend to work with triposes very much in the spirit of internal locale theory. Thus C will usually be a topos and in this case we can cut down the structure needed to specify a C-tripos until it looks very similar to an internal locale.

So let E be a topos and P an E-tripos. Firstly one can replace (c) of Definition 1.1 by:

(c') There is a <u>generic predicate</u> $\sigma \in P(\Sigma)$ (some object Σ of E), such that given any $\varphi \in P(I)$ there is a map $\{\varphi\}: I \longrightarrow \Sigma$ in E with $P\{\varphi\}\sigma \dashv\vdash_I \varphi$.

The membership predicates are then given by taking PI to be Σ^I and $\in_I$ to be $ev_I(\sigma)\in P(I\times\Sigma^I)$, where $ev_I: I\times\Sigma^I \longrightarrow \Sigma$ is the evaluation map. Furthermore, pre-ordering each $E(I,\Sigma)$ by

$$f \vdash_I g \text{ in } E(I,\Sigma) \text{ iff } f(\sigma) \vdash_I g(\sigma) \text{ in } P(I),$$

we make $E(-,\Sigma)$ into an E-indexed category and

$$\begin{aligned} E(I,\Sigma) &\longrightarrow P(I) \\ f &\longmapsto Pf(\sigma) \end{aligned}$$

into an equivalence of E-indexed categories. An E-tripos P, in which each P(I) is actually $E(I,\Sigma)$ for some (fixed) object Σ of E, will be called <u>canonically presented</u>. Now we are interested in triposes in as much as they represent toposes (via the construction to be set out in Chapter 2), and equivalent triposes give equivalent toposes: so when E is a topos (or even just cartesian closed) we can always take an E-tripos to be canonically presented.

In such a tripos, P preserves identities and composition exactly, rather than just up to isomorphism. We can also choose the propositional operations so that they are preserved exactly. For example, letting $m:\Sigma\times\Sigma \longrightarrow \Sigma$ be $\pi_1\wedge\pi_2\in P(\Sigma\times\Sigma)$ (where π_1,π_2 are the projections $\Sigma\times\Sigma\longrightarrow\Sigma$) we have for f,g in $P(I)=E(I,\Sigma)$:

$$\begin{aligned} m\circ\langle f,g\rangle &= P\langle f,g\rangle(\pi_1\wedge\pi_2) \\ &\dashv\vdash_I P\langle f,g\rangle\pi_1 \wedge_I P\langle f,g\rangle\pi_2 \\ &= f\wedge_I g. \end{aligned}$$

Then redefining $\wedge$ on P(I) by $f,g \longmapsto m\circ\langle f,g\rangle$, we have

$$h(f\wedge_I g) = m\circ\langle f,g\rangle\circ h = h(f)\wedge_I h(g),$$

so that each Ph preserves meets exactly.

Can we also choose the quantifiers $\exists,\forall$ so that the Beck conditions hold exactly, rather than just up to isomorphism? This is equivalent to asking that the quantification be computed <u>fibre-wise</u>, i.e. there should be maps $\bigwedge_P, \bigvee_P : (\Omega_E)^\Sigma \longrightarrow \Sigma$ such that

$$(\forall f\varphi)\underline{j} = \bigwedge_P\{\varphi\underline{i} \mid f\underline{i} = \underline{j}\},$$

$$\text{and } (\exists f\varphi)\underline{j} = \bigvee_P\{\varphi\underline{i} \mid f\underline{i} = \underline{j}\},$$

where $I \xrightarrow{f} J$ in C. Proposition 3.14 below gives a charac-
$\varphi \downarrow$
Σ

terisation of those triposes defined over toposes that have fibre-wise quantification. For the moment we note that the triposes defined in 1.4 from locales and in 1.5 from combinatory algebras always have fibre-wise quantification.

To complete the transference of structure from P to Σ, note that the pre-order $\vdash_I$ on $P(I)=E(I,\Sigma)$ can be defined in terms of that on $P(1)$ (1 the terminal object in E). Let $D_P \subseteq E(1,\Sigma)$ consist of those $\varphi : 1 \longrightarrow \Sigma$ with $\top_1 \vdash_1 \varphi$ (the <u>designated truth-values</u>). Then given f,g in P(I),

$$f \vdash_I g \quad \text{iff} \quad PI(\top_1) = \top_I \vdash_I f \to g$$
$$\text{iff} \quad \top_1 \vdash_1 \forall I(f \to g)$$
$$\text{iff} \quad \forall I(f \to g) \in D_P$$

(where I denotes the unique map $I \longrightarrow 1$ in E).

So to summarise, a canonically presented E-tripos with fibre-wise quantification, P, (for E a topos) may be specified by

(a) an object Σ of E,

(b) maps $\to_P : \Sigma\times\Sigma \longrightarrow \Sigma$ and $\bigwedge_P : (\Omega_E)^\Sigma \longrightarrow \Sigma$ in E, and

(c) a subset D_P of $E(1,\Sigma)$,

satisfying various relations (such as those set out immediately before 1.4 of [9]) which ensure that when $\top, \wedge, \bot, \vee, P, \exists, \forall$ and $\in$ are defined as above, the requirements of Definition 1.1 are satisfied.

<u>Remarks</u>

(i) It should be emphasised that what we have here is more general than the definition of a locale. This is because of the two (linked) facts that $\bigwedge_P$ is not necessarily a map assigning greatest lower bounds in Σ and the pre-order $\vdash_I$ on $E(I,\Sigma)$ is not necessarily that induced point-wise by the pre-order $\vdash_1$ on $E(1,\Sigma)$ (i.e. by $\mathbf{D_P}$).

(ii) We cannot necessarily replace the subset $\mathbf{D_P}$ in (c) by a subobject $D \rightarrowtail \Sigma$ (and define : $1 \xrightarrow{p} \Sigma \in \mathbf{D_P}$ iff p factors through D). If this is possible we shall say that P is an internal E-tripos, since the structure and relations needed to specify P are expressible in the internal logic of the topos E (i.e. an internal E-tripos is a model in E of a certain higher-order theory). This is a desirable feature; however to carry out the analysis in Chapter 7, we shall have to consider E-triposes that are not internal. We should regard non-internal triposes in the same way that that we do elementary toposes. We can take the latter to be models of the familiar theory written in a language with two sorts, namely object and arrow sorts; similarly a C-tripos P is a model of a theory written in a three-sorted language, with a sort for the predicates as well as for objects and arrows. This is the view-point of [3] when defining "topos formel". We can interpret the two-sorted theory in the three-sorted one by taking predicates to be subobjects of objects (this is Example 1.3(i) below), and the construction to be given in Chapter 2 reflects the three-sorted theory into the two-sorted one.

Let us turn to some examples of triposes.

1.3 Some special examples

(i) Let E be a topos. The functor $\mathbf{Sub}_E : E^{op} \longrightarrow \mathbf{Cat}$ which assigns to each object I the partially-ordered set of subobjects of I, carries the structure of an E-tripos (the generic predicate being $\top_E : 1 \longrightarrow \Omega_E$).

(ii) Let $\mathbb{T}$ be a theory in a many-sorted, higher-order language

$\mathbb{L}$. Let C be the category with objects the types of $\mathbb{L}$ and arrows the terms of $\mathbb{L}$. Then there is a C-tripos P, with for each type X of $\mathbb{L}$

$P(X)$ = formulae of $\mathbb{L}$ with free variable of type X,

pre-ordered by $\mathbb{T}$-provability (c.f. [3]).

1.4 Localic examples

Let H be an internal locale of a topos E. We define the canonical E-tripos of H, P, to be $\mathbb{E}(-,H)$ with the Heyting algebra structure on each $PI = \mathbb{E}(I,H)$ induced by the internal structure on H; quantification is given fibre-wise by the internal inf map $\bigwedge_H : (\Omega_E)^H \longrightarrow H$; and there is just one designated truth-value in D_P, namely the top element $\top_H : 1 \longrightarrow H$.

Remark

If $\Phi \rightarrowtail H$ is a filter on H, we can modify P by taking $D_P = \{1 \xrightarrow{h} H \mid h \text{ factors through } \Phi \rightarrowtail H\}$ and still get a tripos. More generally if P is a C-tripos and $\Phi \subseteq P(1)$ is a filter, we can redefine the pre-order on each P(I) by

$$\varphi \vdash_I \psi \quad \text{iff} \quad \forall I(\varphi \rightarrow \psi) \in \Phi$$

and get a new C-tripos, which we shall denote P_Φ.

1.5 Realizability examples

To give these examples we need some "combinatory logic". By a (partial) combinatory algebra, $\mathbb{A}$, we mean a set A equipped with a partial binary operation (called application and denoted by $a,b \longmapsto a(b)$) together with elements K and S of A, satisfying for all $a,b,c \in A$ that

$$K(a)(b) \equiv a,$$

$$E(S(a)(b)),$$

$$\text{and } S(a)(b)(c) \equiv a(c)(b(c)),$$

where "E" means "is defined" and $\equiv$ means "one side is defined iff the other is and then they are equal". Since we are dealing with a partial operation, as the notation suggests we are thinking of the theory of combinatory algebras as formulated in D.S. Scott's logic of Identity and Existence set out in [16]; the language has variables x,y,z,..., constants K,S (possibly more) which exist (i.e. EK,ES are axioms) and a function symbol for application which is to be strict (i.e. $Ex(y) \to Ex \wedge Ey$ is an axiom). The terms of the language are usually called <u>combinatory terms</u>. We can introduce <u>λ-terms</u> as an abbreviation for certain combinatory terms in the usual way: if α is a combinatory term, $\lambda x.\alpha$ is another and is defined inductively on the complexity of α as follows:

- if α is x, $\lambda x.\alpha$ is S(K)(K),
- if α is y (different from x), $\lambda x.\alpha$ is K(y),
- if α is c (a constant), $\lambda x.\alpha$ is K(c),
- if α is $\beta_1(\beta_2)$, $\lambda x.\alpha$ is $S(\lambda x.\beta_1)(\lambda x.\beta_2)$.

We thus have that x does not occur in $\lambda x.\alpha$ and the rule of β-conversion holds: $(\lambda x.\alpha)x \equiv \alpha$. Note also that if α has free-variables amongst $x,\vec{x}$ then $E\vec{x} \to E(\lambda x.\alpha)$, so that closed λ-terms are always defined.

The theory of combinatory algebras is of course intended to formalise the notion of (untyped) intensional functions or "rules" applied to one another. There are many models of the theory, the principal being $\mathbb{N}$, the set of natural numbers with application

$$n(m) \equiv \text{value at } m \text{ of the partial recursive function with index } n.$$

Again, any model of the λ-calculus (such as Scott's "graph" model) gives a combinatory algebra, and in these cases application is totally defined.

Now let us see how to make triposes from combinatory algebras. Let $\mathbb{A}$ be a combinatory algebra in a topos E and define a binary operation $\to_{\mathbb{A}}$ on PA the powerobject of A, by

$$p \to_{\mathbb{A}} q = \{ a \mid \forall b \in p(Ea(b) \wedge a(b) \in q) \}.$$

(Since membership is "strict", we shall often supress the Ea(b) in expressions such as the above.) If we think of the elements of PA as "propositions" and the elements of A as "proofs", with $a \in p$ read as "a proves p", then the definition of $\to_{\mathbb{A}}$ models the realizability interpretation of intuitionistic implication.

Define a map $\bigwedge_{\mathbb{A}}: P(PA) \longrightarrow PA$ by

$$\bigwedge_{\mathbb{A}} \Phi = \{ a \mid \forall b \in A, p \subseteq A(p \in \Phi \to a(b) \in p \}.$$

Then on choosing an appropriate set of designated truth-values $\mathbf{D} \subseteq E(1,PA)$, and using the definability results of 1.2, we find that E(-,PA) can be made into a canonically presented E-tripos. There are several possibilities for **D**, but we shall be concerned with two in particular:

(i) Let **D** be induced by the subobject $\{p \mid \exists a \in A(a \in p)\} \rightarrowtail PA$. This makes E(-,PA) into an <u>internal</u> E-tripos (recall Remark (ii) after 1.2).

(ii) Suppose **B** is a subalgebra of $E(1,\mathbb{A})$ (i.e. $\mathbf{B} \subseteq E(1,A)$ contains K and S and is closed under application); then take for **D** all those $1 \longrightarrow PA$ corresponding to subobjects $A' \rightarrowtail A$ through which some $b: 1 \longrightarrow A$ in **B** factors. In general the resulting E-tripos will not be internal, unless **B** is induced by an internal subalgebra of $\mathbb{A}$.

Recalling the Remark after 1.4, note that if **P** denotes an E-tripos obtained from $\mathbb{A}$ as in (ii), then the inhabited subobjects of A form a filter $\Phi \subseteq P(1)$ and P_{Φ} is the tripos obtained from $\mathbb{A}$ as in (i).

<u>Definitions</u>

We shall call a tripos obtained from a combinatory algebra $\mathbb{A}$ by using $\to_{\mathbb{A}}, \bigwedge_{\mathbb{A}}$ and some **D**, a <u>realizability tripos</u> on $\mathbb{A}$. When E is Set and $\mathbb{A}$ is $\mathbb{N}$, (i) and (ii) coincide: we shall call the resulting tripos the <u>effective tripos</u>. (The reason for this name stems from the fact that in the corresponding topos the finite types over the natural numbers are built from the "hereditarily extensional effective operations" in a very simple way: see [8]).

<u>Remark</u>

If P is a realizability tripos on $\mathbb{A}$, the definable structure $\top, \wedge, \bot, \vee, \exists, \forall$ can be given quite simply. For $\top_{\mathbb{A}}, \bot_{\mathbb{A}}$ we take the top and bottom subobjects of A respectively. For $\wedge$, $\vee$ we take

$$p \wedge_{\mathbb{A}} q = \{ a \mid P_0(a) \in p \wedge P_1(a) \in q \}$$

$$p \vee_{\mathbb{A}} q = \{ a \mid (P_0(a) = K \wedge P_1(a) \in p) \vee (P_0(a) = S \wedge P_1(a) \in q) \}$$

where P_0, P_1 are unpairing combinators (for example, $P_0 = \lambda x.x(K)$, $P_1 = \lambda x.x(K(\lambda y.y))$, corresponding to the pairing combinator $P = \lambda xyz.z(x)(y)$; thus $P_0(P(a)(b)) = a$, $P_1(P(a)(b)) = b$. We shall usually write pairing and unpairing as $\langle x,y \rangle$ and $(x)_0, (x)_1$ respectively). If $f: I \longrightarrow J$, then we can take $\exists f: E(I,PA) \longrightarrow E(J,PA)$ so that $\varphi: I \longrightarrow PA$ is sent to

$$(\exists f)\varphi : j \longmapsto \bigcup \{ \varphi i \mid fi = j \}.$$

If f is epi we obtain $\forall f$ by replacing $\bigcup$ by $\bigcap$ in the above; however in general, as the definition of $\bigwedge_{\mathbb{A}}$ suggests, we must define

$$(\forall f)\varphi : j \longmapsto \bigcap \{ [\![fi = j]\!] \rightarrow_{\mathbb{A}} \varphi i \mid i \in I \}$$

where $[\![fi = j]\!] = \{ a \mid fi = j \}$.

<u>1.6 Further examples</u>

The results on iteration in Chapter 6 will provide us with a means for combining the examples in 1.4 and 1.5 to obtain new triposes. For the moment we record an interesting tripos which can in fact be obtained in this way (see 6.4).

Take a realizability tripos, for definiteness let us say the effective tripos, and modify it in the following way. Instead of taking the "propositions" to be subsets of $\mathbb{N}$, let them be subsets together with an equivalence relation. Thus put

$$\Sigma = \{ R \subseteq \mathbb{N} \times \mathbb{N} \mid R \text{ is symmetric and transitive } \}$$

$$\mathsf{R} = \mathrm{Set}(-,\Sigma)$$

and, given R,S in Σ define

$$R \to_{\mathsf{R}} S = \{ (m,n) \mid \forall x,y \in \mathbb{N}\ (x,y) \in R \to (m(x),n(y)) \in S \}.$$

Also, if $\Phi \subseteq \Sigma$ define

$$\bigwedge_{\mathsf{R}} \Phi = \{ (m,n) \mid \forall R \in \Sigma\, \forall x,y \in \mathbb{N} (R \in \Phi \to (m(x),n(y)) \in R) \}.$$

Then with $\mathsf{D} = \{ R \in \Sigma \mid \exists n\, (n,n) \in R \}$, we get a Set-tripos. (It

It is interesting to note that Σ comprises the objects of a cartesian closed category in such a way that $R \to_{\mathsf{R}} S$ is actually the exponential of S by R. Given R in Σ, it restricts to an equivalence relation on $uR = \{n \mid (n,n) \in R\}$: let $|R|$ denote the quotient uR/R. Then the arrows $R \longrightarrow S$ in Σ are to be the elements of $|R \to_{\mathsf{R}} S|$. The pre-order induced by D on Σ is just: $R \vdash_1 S$ iff there is a map from R to S in Σ, i.e. iff $S^R = R \to_{\mathsf{R}} S$ has a global section.

The category Σ has a rather rich structure. It is in fact locally cartesian closed, regular and contains a natural number object (namely $\Delta_{\mathbb{N}}$). Contained within it is a copy of the "hereditarily extensional effective operations" (c.f. Chapter 0).

1.7 Combining examples 1.4 and 1.5

We can weaken the notion of combinatory algebra defined in 1.5 by allowing the partial application to be "many-valued". This has already been suggested by P.Aczel in [1], where he defined the notion of a "D-application" on a complete lattice $(\Sigma,\leqslant)$: this is to be a partial binary operation $x,y \longmapsto x(y)$ on Σ, satisfying amongst other things that there are $K,S \in D$ (an upwards closed subset of Σ) with

$$K(x)(y) \leqslant x$$

and $\quad S(x)(y)(z) \leqslant x(z)(y(z))$.

Since Σ is complete, we get an implication on it by defining

$$x \to y = \bigwedge\{z \mid Ez(x) \wedge z(x) \leqslant y\};$$

and using $\to$, the inf map $\bigwedge$ and $D \subseteq \Sigma$, we get a tripos. As Aczel points out, this construction takes in both the localic triposes of 1.4 and the realizability triposes of 1.5. For the former, take $\Sigma = H$ with D a filter and application just binary meet, so that $K = S = \mathbf{T}_H$; for the latter, take $\Sigma = PA$ with application given by:

$Ep(q)$ iff for all $a \in p$ and $b \in q$ $Ea(b)$, and in this case $p(q) = \{a(b) \mid a \in p \text{ and } b \in q\}$

(see also 4.9(iv) below). Example 1.6 is also a tripos of this kind, as is the result of the iteration to be given in 6.5.

Instead of assuming that $(\Sigma, \leqslant)$ is a <u>complete</u> lattice, we can introduce the notion of covers. Thus suppose we have a set A equipped with a binary relation $\rhd$, a partial binary operation $a, b \longmapsto a(b)$ (called application), elements $K, S \in A$, a subset $D \subseteq A$ (of designated constructions) and a map $C: A \longrightarrow P(PA)$ (assigning covers). We take as axioms the following (where $\beta \trianglerighteq \alpha$ means $(E\alpha \to \beta \rhd \alpha)$, so that $E\alpha \wedge \beta \trianglerighteq \alpha \to E\beta$ holds):

(I) $\quad \rhd$ pre-orders A;

(II)(i) if $a \rhd a'$ and $b \rhd b'$, then $a(b) \trianglerighteq a'(b')$;

(ii) $K(a)(b) \trianglerighteq a$;

(iii) $ES(a)(b)$ and $S(a)(b)(c) \trianglerighteq a(c)(b(c))$;

(III)(i) if $a, a' \in D$ and $Ea(a')$, then $a(a') \in D$;

(ii) $K, S \in D$;

(IV)(i) if $b \in S \in C(a)$, then $b \rhd a$;

(ii) if $S \in C(a)$ and $(\forall s \in S\ Es(b))$, then $Ea(b)$ and $\{s(b) \mid s \in S\}$ is in $C(a(b))$.

Given such a structure, just as for covers on semilattices (c.f. [11]), we can define $I \subseteq A$ to be a <u>C-ideal</u> iff it is downwards closed ($a \rhd b \in I \Rightarrow a \in I$) and closed under covers ($I \supseteq S \in C(a) \Rightarrow a \in I$). Let Σ be the set of C-ideals. For $I, J \in \Sigma$ we define $I \to J \in \Sigma$ just as for realizability triposes:

$$I \to J = \{a \mid \forall b \in I \; \exists c \in J \;\; a(b) = c\}.$$

Similarly for $\Phi \subseteq \Sigma$, we define $\bigwedge \Phi \in \Sigma$ to be

$$\bigwedge \Phi = \{a \mid \forall b \forall I \in \Phi \; \exists c \in I \;\; a(b) = c\},$$

and $\mathbf{D} \subseteq \Sigma$ to be

$$\mathbf{D} = \{I \mid \exists \, a \in D(a \in I)\}.$$

Then using the axioms given above, one may verify that $\to, \bigwedge$ and $\mathbf{D}$ endow Σ with a tripos structure.

1.8 A non-example

Suppose that H is a Heyting algebra (not necessarily a complete one). For a set I, let $\mathbf{P}(I)$ be the sub-Heyting algebra of H^I comprising those $\varphi : I \longrightarrow H$ with $\mathrm{im}(\varphi)$ <u>finite</u>. Since if $\varphi \in \mathbf{P}(I)$ and $f : J \longrightarrow I$ then $\varphi \circ f \in \mathbf{P}(J)$, $\mathbf{P}$ is a Set-indexed category. Moreover it is not hard to see that it satisfies (a) and (b) of Definition 1.1. But so long as H is itself infinite, there can be no generic predicate for $\mathbf{P}$, which is therefore not a tripos. However in 2.9 we will see, at least in the case that H is Boolean, that this defect is more apparent than real.

2. THE BASIC CONSTRUCTION

As the acronym tripos suggests, from each C-tripos P (C a finitely complete category), one can construct a topos, which will be denoted C[P]. This construction is a direct generalisation of the construction by Higgs and Fourman-Scott [6] of the topos of H-valued sets of a locale H.

Having established the internal logic of a tripos, many of the proofs in this Chapter reduce to straightfoward deductions which we omit; the reader should refer to [9] for more details.

2.1 P-interpretation of languages

Suppose we are given a many-sorted first-order language without equality, $\mathbb{L}$ (c.f. [13], for example) and a C-tripos P.

A P-interpretation of $\mathbb{L}$ is given by the following data:

(a) for each type (sort) $\underline{X}$ of $\mathbb{L}$, an object X of C;

(b) for each function symbol $\underline{f}$ of $\mathbb{L}$, of signature $(\underline{X}_1,\ldots,\underline{X}_n;\underline{Y})$ say, an arrow $f:\prod_i X_i \longrightarrow Y$ in C;

(c) for each relation symbol $\underline{R}$ of $\mathbb{L}$, of signature $(\underline{X}_1,\ldots,\underline{X}_n)$ say, an element R of $P(\prod_i X_i)$.

The obvious inductive definitions then allow us to interpret terms of $\mathbb{L}$ as arrows in C and formulae of $\mathbb{L}$ as elements of the P(I). Thus given a formula φ whose free variables are amongst $\vec{\underline{x}} = (\underline{x}_1,\ldots,\underline{x}_n)$, we assign an interpretation

$$[\![\varphi(\vec{\underline{x}})]\!] \in P(\prod_i X_i)$$

(where $\underline{X}_i$ is the type of $\underline{x}_i$); for example

$$[\![\forall \underline{x}_2(\varphi(\underline{x}_1)\wedge\psi(\underline{x}_2))]\!] = \forall\pi_1(P\pi_1[\![\varphi(\underline{x}_1)]\!]\wedge P\pi_2[\![\psi(\underline{x}_2)]\!]),$$

where $\pi_i:X_1\times X_2 \longrightarrow X_i$ are the projections.

Given a finite sequence Γ of formulae of $\mathbb{L}$, a formula φ of $\mathbb{L}$ and a string of variables $\vec{\underline{x}}$ containing the free variables of $\Gamma\cup\{\varphi\}$, we have a semantic entailment notion

$$\Gamma \models^{P}_{\vec{\underline{x}}} \varphi$$

which holds precisely when

$$\bigwedge_{\gamma\in\Gamma} [\![\gamma(\vec{\underline{x}})]\!] \vdash [\![\varphi(\vec{\underline{x}})]\!] \text{ in } P(\prod_i X_i)$$

We shall say that $\varphi(\vec{\underline{x}})$ is <u>valid</u> for the P-interpretation iff $\emptyset \models^{P}_{\vec{\underline{x}}} \varphi$ (usually written $P \models_{\vec{\underline{x}}} \varphi$). A <u>P-model</u> of a theory $\mathbb{T}$ in the language $\mathbb{L}$ is then a P-interpretation that makes all the axioms of $\mathbb{T}$ valid.

Let $\Gamma \vdash_{\vec{\underline{x}}} \varphi$

denote the "labelled" syntactic entailment notion of intuitionistic logic (a detailed description of which can be found in [2]). Then we have the following fundamental result, whose proof is by induction over the definition of $\vdash_{\vec{\underline{x}}}$:

<u>Soundness Lemma</u>

If $\Gamma \vdash_{\vec{\underline{x}}} \varphi$ holds, then so does $\Gamma \models^{P}_{\vec{\underline{x}}} \varphi$, for any P-interpretation of $\mathbb{L}$. □

2.2 Objects, relations and functions

By a <u>P-object</u> (X,=) we shall mean a P-model of the theory of eqality (of partial elements; see [16]). Thus X is an object of C and =, a predicate in $P(X\times X)$, satisfies

$$P \models \underline{x} = \underline{x}' \rightarrow \underline{x}' = \underline{x},$$

and $P \models \underline{x} = \underline{x}' \wedge \underline{x}' = \underline{x}'' \rightarrow \underline{x} = \underline{x}''$.

The predicate $E_X = [\![\underline{x} = \underline{x}]\!]$ in P(X) is thus the "extent to which $\underline{x}$ exists"; we shall usually write $[\![\underline{x} \in X]\!]$ for E_X.

A <u>relation</u> on P-objects $(X_1,=_1),\dots,(X_n,=_n)$ is an element R of $P(\prod_i X_i)$ which respects the equalities, i.e.

$$P \models R(\underline{x}_1,\dots,\underline{x}_n) \wedge \bigwedge_i \underline{x} =_i \underline{x}_i' \rightarrow R(\underline{x}_1',\dots,\underline{x}_n').$$

Say that R is <u>strict</u> iff in addition we have

$$P \models R(\underline{x}_1,\dots,\underline{x}_n) \rightarrow \bigwedge_i \underline{x}_i \in X_i.$$

A strict relation F on P-objects (X,=),(Y,=) will be called <u>functional</u> iff it satisfies

(a) F is <u>single-valued</u>, i.e. $P \models F(\underline{x},\underline{y}) \wedge F(\underline{x},\underline{y}') \rightarrow \underline{y} = \underline{y}'$;

(b) F is <u>total</u>, i.e. $P \models \underline{x} \in X \rightarrow \exists \underline{y} F(\underline{x},\underline{y})$.

<u>Remark</u>

If F and F' are both functional relations from (X,=) to (Y,=), to prove that $F \dashv\vdash F'$, we need only show that $F \vdash F'$.

<u>2.3 Theorem</u>

Let **P** be a C-tripos. There is a category, denoted C[**P**], whose objects are the **P**-objects (X,=) and whose arrows from (X,=) to (Y,=) are $\dashv\vdash$ equivalence classes of functional relations from (X,=) to (Y,=). (If $f:(X,=) \longrightarrow (Y,=)$ is a typical arrow in C[**P**], a choice of functional relation representing f will usually be denoted by the corresponding upper-case letter, F.) The identity arrow on (X,=) is the equivalence class of =, and the composite of $f:(X,=) \longrightarrow (Y,=)$ and $g:(Y,=) \longrightarrow (Z,=)$ is represented by $[\![\exists_{Y} \underline{y} (F(\underline{x},\underline{y}) \wedge G(\underline{y},\underline{z}))]\!]$ in $P(X \times Z)$.

Moreover C[**P**] is a topos.

□

<u>Remark</u>

If **P** is a C-indexed category which only satisfies parts (a) and (b) of Definition 1.1, we can still carry out the construction of the category C[**P**]. It will be a "logos" in the sense of [3], i.e. a regular category with stable finite sups, Heyting implication and universal quantification of subobjects along maps. Note however that we do not claim that part (c) of 1.1 is a <u>necessary</u> condition for C[**P**] to be a topos: see 2.9.

In the next few paragraphs we prove 2.3 by giving the structure of C[**P**] in detail.

2.4 Finite limits in C[P]

(i) Let 1 be the terminal object in C. Then $(1, T_{1\times 1})$ is terminal in C[P].

(ii) Given

$$\begin{array}{ccc} & & (X_2,=_2) \\ & & \big\downarrow f_2 \\ (X_1,=_1) & \xrightarrow{f_1} & (Y,=) \end{array}$$

in C[P], we can consruct the pullback as

$$\begin{array}{ccc} (X_1\times X_2,=_P) & \xrightarrow{p_2} & (X_2,=_2), \\ \big\downarrow p_1 & & \\ (X_1,=_1) & & \end{array}$$

with

$$[\![\underline{a} =_P \underline{a}']\!] = [\![(\pi_1\underline{a} =_1 \pi_1\underline{a}') \wedge (\pi_2\underline{a} =_2 \pi_2\underline{a}') \wedge \underline{a} \in P]\!]$$

in $P((X_1\times X_2)\times(X_1\times X_2))$, where

$$[\![a \in P]\!] = [\![\exists \underline{y}\, (F_1(\pi_1\underline{a} = \underline{y}) \wedge F_2(\pi_2\underline{a} = \underline{y}))]\!]$$

in $P(X_1\times X_2)$ and p_i is represented by $[\![\underline{a} \in P \wedge (\pi_i\underline{a} =_i \underline{x}_i)]\!]$ in $P((X_1\times X_2)\times X_i)$.

(iii) Taking (Y,=) to be the terminal object, we see from (ii) that the product of $(X_1,=_1)$ and $(X_2,=_2)$ in C[P] is $(X_1\times X_2,=_{12})$, where $[\![\underline{a} =_{12} \underline{a}']\!] = [\![(\pi_1\underline{a} =_1 \pi_1\underline{a}') \wedge (\pi_2\underline{a} =_2 \pi_2\underline{a}')]\!]$.

2.5 Monomorphisms, epimorphisms and isomorphisms in C[P]

(i) Given a commutative square

$$\begin{array}{ccc} (P,=) & \xrightarrow{h} & (Y,=) \\ \big\downarrow k & & \big\downarrow g \\ (X,=) & \xrightarrow{f} & (Z,=) \end{array}$$

in C[P], it is a pullback iff for any (and hence all) representatives F,G,H,K of f,g,h,k we have

$$P \models F(\underline{x},\underline{z}) \wedge G(\underline{y},\underline{z}) \rightarrow \exists \underline{p}\, (H(\underline{p},\underline{y}) \wedge K(\underline{p},\underline{x})),$$

and $P \models H(\underline{p},\underline{y}) \wedge K(\underline{p},\underline{x}) \wedge H(\underline{p}',\underline{y}) \wedge K(\underline{p}',\underline{x}) \rightarrow \underline{p} = \underline{p}'$.

(ii) As a corollary of (i), we have that $f:(X,=) \longrightarrow (Z,=)$ is mono in C[P] iff

$$P \models F(\underline{x},\underline{z}) \wedge F(\underline{x}',\underline{z}) \rightarrow \underline{x} = \underline{x}' \text{ (i.e. single-valued in } \underline{x}).$$

(iii) Similarly, f is epi in C[P] iff

$$P \models \underline{z} \in Z \ \rightarrow \ \exists \underline{x} F(\underline{x},\underline{z}) \quad \text{(i.e. total in } \underline{z}\text{)}.$$

(iv) Putting (ii) and (iii) together, f is an isomorphism iff F is a functional relation both from (X,=) to (Z,=) and vice versa.

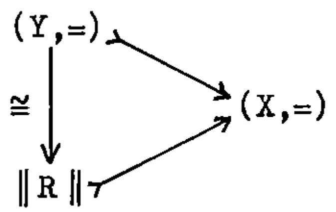
2.6 Subobjects, powerobjects and partial maps in C[P]

(i) Let (X,=) be a P-object and R a strict relation on (X,=). Define a new P-object $\|R\|$, by changing the equality on X to

$$[\![\underline{x} = \underline{x}' \wedge R(\underline{x})]\!].$$

Then by 2.5(i), the functional relation $[\![R(\underline{x}) \wedge \underline{x} = \underline{x}']\!]$ represents a monomorphism $\|R\| \rightarrowtail (X,=)$. We shall call such monics canonical.

Given any monic $m: (Y,=) \rightarrowtail (X,=)$, we get a strict relation on (X,=) by defining $R = [\![\exists \underline{y} M(\underline{y},\underline{x})]\!]$. Then by 2.5(iii), M represents an isomorphism $(Y,=) \longrightarrow \|R\|$, and furthermore the triangle



commutes. Thus every subobject of (X,=) in C[P] may be represented by a canonical monic.

If R,S are strict relations on (X,=), then

$$R \vdash_X S \quad \text{iff} \quad \|R\| \leqslant \|S\| \text{ in Sub}(X,=).$$

Thus the map

$$\{R \in (X) \mid R \text{ a strict relation for } (X,=)\} \longrightarrow \text{Sub}(X,=)$$
$$R \longmapsto \|R\| \rightarrowtail (X,=)$$

is an equivalence of pre-ordered sets. We shall have more to say about this equivalence in 2.10. For the moment, note that the pull-back of $\|S\| \rightarrowtail (Y,=)$ along $f: (X,=) \longrightarrow (Y,=)$ is represented by the canonical monic

$$\| \exists \underline{y} \big(F(\underline{x},\underline{y}) \wedge S(\underline{y}) \big) \| \rightarrowtail (X,=).$$

(ii) Given a P-object $(X,=)$, its powerobject $P(X,=)$ has underlying object PX (given to us by part (c) of Definition 1.1) and equality

$$[\![\underline{R} =_{PX} \underline{S}]\!] = [\![\forall \underline{x}(\underline{x} \in_X \underline{R} \leftrightarrow \underline{x} \in_X \underline{S}) \wedge \underline{R} \in PX]\!],$$

where $[\![\underline{R} \in PX]\!] = [\![\forall \underline{x},\underline{x}'(\underline{x} \in_X \underline{R} \wedge \underline{x}=\underline{x}' \rightarrow \underline{x}' \in_X \underline{R}) \wedge \forall \underline{x}(\underline{x} \in_X \underline{R} \rightarrow \underline{x} \in X)]\!]$

(i.e. $\underline{R}$ exists to the extent that it is a strict relation for $(X,=)$). Here $\in_X$ is the membership predicate on X in $P(X \times PX)$.

The membership relation $\in_{(X,=)} \rightarrowtail (X,=) \times P(X,=)$ for $(X,=)$ in $C[P]$ is given by the canonical monic

$$\| \underline{R} \in PX \wedge \underline{x} \in_X \underline{R} \| \rightarrowtail (X \times PX,=).$$

(iii) We shall need (in Chapter 3) to use partial map classifiers in the topos $C[P]$, which we now consruct explicitly from powerobjects.

Given a partial map between P-objects $(X,=)$ and $(Y,=)$, say

$$\begin{array}{ccc} \circ & \xrightarrow{\ f\ } & (Y,=) \\ {\scriptstyle m}\big\downarrow & & \\ (X,=) & & \end{array}$$

we can (by 2.6(i)) assume that the monomorphism m is canonical, determined by some strict relation R for $(X,=)$. Hence we can pick some F in $P(X \times Y)$ representing f, and F will be a <u>partial functional relation</u> from $(X,=)$ to $(Y,=)$, i.e. it will be strict and single-valued (in the sense of 2.2). Conversely every such relation determines a partial map $(X,=) \rightharpoonup (Y,=)$ in $C[P]$ (with $R = [\![\exists \underline{y} F(\underline{x},\underline{y})]\!]$).

Now, given a P-object $(X,=)$, the predicate

$$[\![R \in \widetilde{X}]\!] = [\![\forall \underline{x},\underline{x}'(\underline{x} \in_X \underline{R} \wedge \underline{x}' \in_X \underline{R} \leftrightarrow \underline{x} \in_X \underline{R} \wedge \underline{x}=\underline{x}')]\!]$$

in $P(PX)$ is a strict relation for $P(X,=)$ and so determines a canonical monic, which we denote by

$$\widetilde{(X,=)} \rightarrowtail P(X,=).$$

Then $[\![\, \underline{x}\in_X\underline{R} \wedge \underline{R}\in\breve{X} \,]\!]$ in $P(X\times PX)$ represents a map $\eta:(X,=)\longrightarrow\widetilde{(X,=)}$ which by 2.5(ii) is mono. This is the partial map classifier for $(X,=)$. Given a partial functional relation F from $(Y,=)$ to $(X,=)$,

$$[\![\, \underline{y}\in Y \wedge \forall\underline{x}(\underline{x}\in_X\underline{R} \leftrightarrow F(\underline{x},\underline{y})) \,]\!]$$

in $P(Y\times PX)$ represents a map $(Y,=)\longrightarrow\widetilde{(X,=)}$ making

$$\begin{array}{ccc} \|R\| & \longrightarrow & (X,=) \\ \big\downarrow & & \big\downarrow \eta \\ (Y,=) & \longrightarrow & \widetilde{(X,=)} \end{array}$$

a pullback suare, and is the unique such map.

2.7 Change of designated truth-values and filterpowers

Suppose that $\Phi \subseteq P(1)$ is a filter. Recall from the Remark after 1.4 that changing D_P to Φ gives a new C-tripos, P_Φ. Now by 2.6(i) each $\varphi\in\Phi$ gives a canonical monic $\|\varphi\| \rightarrowtail 1$ in $C[P]$: in this way we obtain a filter of subobjects of 1 in $C[P]$, which we shall denote by $\hat{\Phi}$. It is not hard to see that $C[P_\Phi]$ is actually equivalent to the filterpower topos $C[P]_{\hat{\Phi}}$ (c.f. 9.3 of [10]). Under this equivalence, the logical functor $C[P]\longrightarrow C[P]_{\hat{\Phi}}$ is identified with the obvious functor from $C[P]$ to $C[P_\Phi]$ which is just the identity on objects.

2.8 The examples revisited

Let us apply the construction $C, P \longmapsto C[P]$ to the examples in 1.3, 1.4 and 1.5.

1.3(i) $E[\mathbf{Sub}_E]$ is of course equivalent to E (the equivalence being a very special case of Proposition 3.8 below, with $\Delta: C\longrightarrow E$ equal to id_E).

1.3(ii) $C[P]$ is equivalent to the "syntactic category" of $\mathbb{T}$ as defined in [3] or [4].

1.4 Just as in [6], we have $E[P] \simeq E[H]$, the topos of

E-valued sheaves on H. As in 2.6, filters on H give filter-powers of E[H].

1.5 These give the <u>realizability toposes</u> first devised by M. Hyland, whose properties are discussed in [8]. When **P** is the effective tripos, we shall denote Set[**P**] by Eff, the <u>effective topos</u>. Note that Set itself is (equivalent to) a realizability topos, namely that given by the <u>degenerate combinatory algebra **0**</u>, which has just one element K=S.

<u>2.9 The non-example revisited</u>

The Set-indexed category **P** of 1.8 satisfied (a) and (b) but not (c) of Definition 1.1. Thus as in the Remark after Theorem 2.3, we expect Set[**P**] to be only a logos. However in the case that **P** is given by a <u>Boolean</u> algebra B, then Set[**P**] is actually a (Boolean) topos. The powerobject of (X,=) in Set[**P**] is of the form ($\mathbb{P}$X,=) where PX is the powerset of X. Moreover, calculating Set[**P**](1,Ω) we find it is isomorphic (qua Heyting algebra) to the Boolean algebra B. Now P.J.Freyd has shown how, starting from B, one can construct a Boolean topos $\mathfrak{B}$ with algebra of truth-values isomorphic to B. We may describe $\mathfrak{B}$ as follows (c.f. Exercise 9.11 of [10] and also [17]):

Let $\mathfrak{F}$ be the filtered poset of finite Boolean subalgebras of B. Every $F \in \mathfrak{F}$ is atomic: $F \cong 2^{aF}$ where $aF = \{\text{atoms of } F\}$; also every inclusion $F \subseteq G$ in $\mathfrak{F}$ is induced by a (surjective) map $aG \longrightarrow aF$. Thus $F \longmapsto Sh(F) \simeq Set^{aF}$ gives a filtered diagram of Boolean toposes and logical functors, and $\mathfrak{B}$ is defined to be the colimit of this diagram. Filtered colimits in the category of (Boolean) toposes and logical functors are created by the forgetful functor to the category of categories and functors.

Thus $\mathfrak{B}(1,\Omega) \cong \mathrm{colim}_{F\in\mathfrak{F}} Sh(F)(1,\Omega) \cong \mathrm{colim}_{F\in\mathfrak{F}} F \cong B$ in the category of Boolean algebras.

Now given $F\in\mathfrak{F}$, if $(X,=)$ is an F-valued set it is also a $\mathbf{P}$-object, since $F \subseteq B$ and F is finite. In this way we get a functor $Sh(F) \longrightarrow Set[\mathbf{P}]$ which is logical; and as F varies over $\mathfrak{F}$ this gives a cone under the diagram $F \longmapsto Sh(F)$. It is in fact a colimiting cone, so that <u>Set[P] is eqivalent to $\mathfrak{B}$</u>.

In the definition of $\mathbf{P}$, it is natural to consider limiting the types to <u>finite</u> sets, i.e. to restrict $\mathbf{P}$ to a Set_f-indexed category with $\mathbf{P}(I) = Set(I,B)$ for I an object in Set_f, the category of finite sets and maps. Once again, $Set_f[\mathbf{P}]$ is a topos. We may describe a very simple category equivalent to $Set_f[\mathbf{P}]$ as follows:

> The objects are finite sequences $b_1,\ldots,b_n$ of elements of B; the arrows $f:(b_1,\ldots,b_n) \longrightarrow (b'_1,\ldots,b'_m)$ are $n\times m$ matrices (f_{ij}) of elements of B such that for each $i=1,\ldots,n$ $f_{i1},\ldots,f_{im}$ is a partition of b_i with $f_{ij} \leq b'_j$ each $j=1,\ldots,m$.

(In the terminology of Chapter 3, what we have described is the full subcategory of $Set_f[\mathbf{P}]$ comprising the subconstant $\mathbf{P}$-objects, which since B is Boolean is equivalent to the whole of $Set_f[\mathbf{P}]$.)

<u>2.10 Internal and external logic</u>

What advantage is it to know that some topos E is actually $C[\mathbf{P}]$ for a C-tripos $\mathbf{P}$? One answer is that then we have a presentation of the internal logic of E externally as the logic of $\mathbf{P}$. For in 2.6 we saw that $Sub(X,=)$ was equivalent to a sub-preordered set of $\mathbf{P}(X)$. Using this equivalence we may identify the logical operations on subobjects in E with operations on the predicates in $\mathbf{P}$. We have already done this in 2.6 for substitution along a map. The propositional operations are straightfoward, although we must remember to keep all relations strict. Thus

$$\|R\| \cap \|R'\| \cong \| R \wedge R' \|,$$
$$\|R\| \cup \|R'\| \cong \| R \vee R' \|,$$
$$\|R\| \rightarrow \|R'\| \cong \| (R(\underline{x}) \rightarrow R(\underline{x}')) \wedge \underline{x} \in X \|,$$
$$1 \cong \| \underline{x} \in X \|,$$
and
$$0 \cong \| \bot_X \|,$$

in Sub(X,=). Similarly, given $f:(X,=) \longrightarrow (Y,=)$ and R a strict relation for (X,=), we find that

$$\exists_f \|R\| \cong \| \exists \underline{x} (F(\underline{x},\underline{y}) \wedge R(\underline{x})) \|,$$
and
$$\forall_f \|R\| \cong \| \forall \underline{x} (F(\underline{x},\underline{y}) \rightarrow R(\underline{x})) \wedge \underline{y} \in Y \|$$

in Sub(Y,=).

In general, suppose we have a many-sorted, first-order languge with equality $\mathbb{L}$, and an interpretation of $\mathbb{L}$ in the topos C[P] in the usual sense of categorical logic. Thus each type $\underline{X}$ of $\mathbb{L}$ is interpreted as some P-object (X,=), each function symbol $\underline{f}$, of signature $(\underline{X}_1,\ldots,\underline{X}_n;\underline{Y})$, by a map $f:\prod(X_i,=_i) \longrightarrow (Y,=)$ represented by a functional relation $F \in P(\prod X_i \times Y)$ say, and each relation symbol $\underline{R}$, of signature $(\underline{X}_1,\ldots,\underline{X}_n)$, by a subobject $\cdot\!\!\rightarrowtail \prod(X_i,=_i)$ canonically represented by some relation $R \in P(\prod X_i)$.

These choices determine a P-interpretation of a new language $\mathbb{L}^*$ without equality: $\mathbb{L}^*$ has the same types as $\mathbb{L}$, relation symbols $=_X, \in_X$ (of signatures $(\underline{X},\underline{X})$ and $(\underline{X})$) for each type $\underline{X}$, relation symbols $\underline{R}$ for each relation symbol $\underline{R}$ of $\mathbb{L}$ (with the same signature), and relation symbols $\underline{F}$ (of signature $(\underline{X}_1,\ldots,\underline{X}_n,\underline{Y})$) for each function symbol $\underline{f}$ (of signature $(\underline{X}_1,\ldots,\underline{X}_n;\underline{Y})$) of $\mathbb{L}$.

Now we may translate every $\mathbb{L}$-formula φ into a $\mathbb{L}^*$-formula φ^* as follows:

$(\underline{y}=\underline{y})^*$ is $(\underline{y}=\underline{y})$,

$(\underline{f}(t_1,\ldots,t_n)=\underline{y})^*$ is $\exists\, \underline{x}_1,\ldots\underline{x}_n(\underline{F}(\underline{x}_1,\ldots,\underline{x}_n,\underline{y}) \wedge \bigwedge_i (t_i=\underline{x}_i)^*)$,

$P(t_1,\dots,t_n)^*$ is $\exists\, \underline{x}_1,\dots,\underline{x}_n(P(\underline{x}_1,\dots,\underline{x}_n) \wedge \bigwedge_i (t_i{=}\underline{x}_i)^*)$,

(where t_i are terms and P is $\underline{R}$, $=_X$ or $\in_X$)

$(\varphi \wedge \psi)^*$ is $\varphi^* \wedge \psi^*$,

$(\varphi \vee \psi)^*$ is $\varphi^* \vee \psi^*$,

$(\varphi \rightarrow \psi)^*$ is $\varphi^* \rightarrow \psi^*$,

$(\top)^*$ is $\top$,

$(\bot)^*$ is $\bot$,

$(\exists \underline{x}\,\varphi)^*$ is $\exists \underline{x}\,(\underline{x} \in \underline{X} \wedge \varphi^*)$

and $(\forall \underline{x}\varphi)^*$ is $\forall \underline{x}(\underline{x} \in \underline{X} \rightarrow \varphi^*)$.

Then from the remarks above, by induction on the structure of φ we have:

<u>Proposition</u>

The P-interpretation of φ^*, $[\![\varphi^*(\vec{\underline{x}})]\!] \in (\prod X_i)$, is a relation for $\prod(X_i,=_i)$. If we make it strict by forming

$$\varphi^* \wedge \bigwedge_i \underline{x}_i \in \underline{X}_i,$$

then the canonical monic $\| \varphi^* \wedge \bigwedge_i \underline{x}_i \in \underline{X}_i \| \rightarrowtail \prod(X_i,=_i)$ determines the same subobject as $\{\![\varphi(\vec{\underline{x}})]\!\} \rightarrowtail \prod(X_i,=_i)$, the interpretation of φ in C[P] in the usual categorical sense (see [13], for example).

□

In this sense the internal logic of C[P] coincides with the external logic of P. It is apparent from the above (and no suprise since we are generalising the notion of H-valued sets) that the natural logic for interpretations of languages <u>with</u> equality in P is the "higher-order intuitionistic logic of partially defined elements" expounded in [16].

3. THE "CONSTANT OBJECTS" FUNCTOR

We now wish to investigate the properties of toposes of the form C[P]. When P is the canonical E-tripos of some internal locale H (E a topos), we know (2.8) that C[P] is E[H], E-valued sheaves on H. Hence there is a geometric morphism

$$E[H] \longrightarrow E$$

with inverse image functor Δ, assigning constant sheaves, and direct image functor Γ, taking global sections; moreover this E-topos has 1 as an object of generators, i.e. for each sheaf X in E[H] there is an object I in E and maps

$$\begin{array}{ccc} \cdot & \longrightarrow\!\!\!\!\!\rightarrow & X \\ \downarrow & & \\ \Delta(I) & & \end{array}$$

presenting X as a subquotient of a constant object. Conversely every E-topos $f:F \longrightarrow E$ with this property arises (up to equivalence) from an E-locale in this way (namely $F \simeq E[f_*\Omega_F]$).

How much of this remains true for an arbitrary tripos? Certainly not all: for example, in Chapter 5 we shall see that the effective topos is not defined over Set at all, but rather contains Set as a subtopos. However we can still define the analogue of the functor Δ and salvage some of its properties, as we now proceed to show.

3.1 Definition

Let C be a finitely complete category and P a C-tripos. The <u>constant P-object</u> on an object X of C, denoted $\Delta_P(X)$, has underlying object X and equality predicate $[\![\, \underline{x} =_{\Delta X} \underline{x} \,]\!] = \exists\Delta_X(\top_X)$, where $\Delta_X : X \longrightarrow X\ X$ is the diagonal map in C.

For each map $f:X \longrightarrow Y$ in C, $\exists\langle id_X, f\rangle \top_X \in P(X\times Y)$ represents a map $\Delta_P(f):\Delta_P(X) \longrightarrow \Delta_P(Y)$ in $C[P]$. It is easy to verify that this defines a functor

$$\Delta_P : C \longrightarrow C[P].$$

3.2 Remarks

(i) Note that $\Delta_P(f)$ is also represented by $[\![f\underline{x} =_{\Delta Y} \underline{y}]\!]$ in $P(X\times Y)$. For, applying the Beck condition for $\exists$ to the pullback

$$\begin{array}{ccc} X\times Y & \longrightarrow & Y\times Y \\ {\scriptstyle <id_X, f>}\uparrow & & \uparrow{\scriptstyle \Delta_Y} \\ X & \longrightarrow & Y \end{array}$$

we have $P(f\times id)(\exists\Delta_Y)\top_Y \dashv\vdash \exists<id,f>(Pf)\top_Y \dashv\vdash \exists<id,f>\top_X$.

(ii) In the case that P is the canonical E-tripos of an E-locale H, we find that $[\![\underline{x} =_{\Delta X} \underline{x}']\!] = \bigvee_H\{ \top_H \mid \underline{x} = \underline{x}' \}$, as we should hope (c.f. 4.8(iii) of [6]).

In the localic case Δ is left exact, and this remains true in general. To prove it we need the following result about the maps $C/X \longrightarrow P(X)$, sending f to $\exists f(\top_{domf})$:

3.3 Lemma

If

$$\begin{array}{ccc} P & \xrightarrow{h} & Y \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Z \end{array}$$

is a pullback square in C, then we have $(\exists f)\top_X \wedge (\exists g)\top_Y \dashv\vdash (\exists fk)\top_P$ in $P(Z)$.

Proof

$(\exists fk)\top_P \dashv\vdash (\exists f)(\exists k)(Pk)\top_X \vdash (\exists f)\top_X$, and $(\exists fk)\top_P \vdash (\exists g)\top_Y$ similarly; so $(\exists fk)\top_P \vdash (\exists f)\top_X \wedge (\exists g)\top_Y$.

Conversely, $\top_X \vdash (\exists k)(Ph)\top_Y \rightarrow (Pf)(\exists fk)\top_Z$

$\dashv\vdash (Pf)(\exists g)\top_Y \rightarrow (Pf)(\exists fk)\top_Z$ (Beck)

so $(\exists f)\top_X \vdash (\exists g)\top_Y \rightarrow (\exists fk)\top_Z$ (P preserves $\rightarrow$, and $\exists \dashv P$)

i.e. $(\exists f)\top_X \wedge (\exists g)\top_Y \vdash (\exists fk)\top_Z$.

□

3.4 Proposition

The functor $\Delta_P : C \longrightarrow C[P]$ preserves finite limits.

Proof

(i) Δ_P preserves the terminal object 1 of C since $\Delta_1 : 1 \longrightarrow 1\times 1$ is an isomorphism, and hence $(\exists\Delta_1)\top_1 \dashv\vdash \top_{1\times 1}$.

(ii) To see that Δ_P preserves a pullback square

$$\begin{array}{ccc} P & \xrightarrow{h} & Y \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Z \end{array}$$

in C, apply 2.5(i): we need to show that

$$P \models f\underline{x} =_{\Delta Z} \underline{z} \wedge g\underline{y} =_{\Delta Z} \underline{z} \rightarrow \exists \underline{p}(k\underline{p} =_{\Delta X} \underline{x} \wedge h\underline{p} =_{\Delta Y} \underline{y}),$$

and $P \models k\underline{p} =_{\Delta X} \underline{x} \wedge h\underline{p} =_{\Delta Y} \underline{y} \wedge k\underline{p}' =_{\Delta X} \underline{x} \wedge h\underline{p}' =_{\Delta Y} \underline{y} \rightarrow \underline{p} =_{\Delta P} \underline{p}'$,

which follow by applying Lemma 3.3 to various pullback squares.

Thus Δ_P preserves the terminal object and pullbacks, and hence all finite limits. □

We shall now reconstuct the tripos P from the topos C[P] via Δ_P. The following lemma gives a useful formulation of the constant equality on X:

3.5 Lemma

With $=_{\Delta X} \in P(X\times X)$ as in Definition 3.1, we have that

$$P \models \underline{x} =_{\Delta X} \underline{x}' \leftrightarrow \forall \underline{R}(\underline{x} \in_X \underline{R} \leftrightarrow \underline{x}' \in_X \underline{R}).$$

Proof

Since $\top_X \dashv\vdash P\Delta_X [\![\forall \underline{R}(\underline{x} \in_X \underline{R} \leftrightarrow \underline{x}' \in_X \underline{R})]\!]$, we have that $(\exists\Delta_X)\top_X \vdash [\![\forall \underline{R}(\underline{x} \in_X \underline{R} \leftrightarrow \underline{x}' \in_X \underline{R})]\!]$.

Conversely, by (c) of Definition 1.1, we can find $\delta : X \longrightarrow PX$ in C with $P \models \underline{x} \in_X \delta(\underline{x}') \leftrightarrow \underline{x} =_{\Delta X} \underline{x}'$. Then

$$\begin{aligned} \forall \underline{R}(\underline{x} \in_X \underline{R} \leftrightarrow \underline{x}' \in_X \underline{R}) &\models \underline{x} \in_X \delta(\underline{x}') \leftrightarrow \underline{x}' \in_X \delta(\underline{x}') \\ &\models \underline{x} =_{\Delta X} \underline{x}' \leftrightarrow \underline{x}' =_{\Delta X} \underline{x}' \\ &\models \underline{x} =_{\Delta X} \underline{x}' \qquad (\text{as } [\![\underline{x}' =_{\Delta X} \underline{x}']\!] = \top_X). \end{aligned}$$

□

It follows that every element of $P(X)$ is a relation for $\Delta_P X$, and necessarily strict since $[\![\underline{x} =_{\Delta X} \underline{x}]\!] = \top_X$. So from 2.6(i) we have that the map

$$\begin{array}{rccl} \|\cdot\|_X \colon & P(X) & \longrightarrow & \mathrm{Sub}_{C[P]}(\Delta_P X) \\ & R & \longmapsto & \|R\| \rightarrowtail \Delta_P X \end{array}$$

is an equivalence. In fact:

3.6 Proposition

The above maps $\|\cdot\|_X$ make P and $\mathbf{Sub}_{C[P]} \circ \Delta_P^{op}$ into equivalent C-indexed categories.

Proof

Given $f\colon X \longrightarrow Y$ in C and $S \in P(Y)$, recall that the pullback of $\|S\| \rightarrowtail \Delta_P Y$ along $\Delta_P f$ is $\|R\| \rightarrowtail \Delta_P X$ where

$$\begin{array}{rll} R & = [\![\exists \underline{y} (f\underline{x} =_{\Delta Y} \underline{y} \wedge S(\underline{y}))]\!] & \\ & \dashv\vdash [\![\exists \underline{y} S(f\underline{x})]\!] & \text{by Lemma 3.5} \\ & \dashv\vdash Pf(S). & \end{array}$$

Thus

$$\begin{array}{ccc} P(X) & \xrightarrow[\cong]{\|\cdot\|_X} & \mathrm{Sub}(\Delta_P X) \\ \uparrow{\scriptstyle Pf} & \searrow\!\!\nwarrow & \uparrow{\scriptstyle (\Delta_P f)^{-1}} \\ P(Y) & \xrightarrow[\cong]{\|\cdot\|_Y} & \mathrm{Sub}(\Delta_P Y) \end{array}$$

commutes up to isomorphism. □

Remark

Since the first-order structure on P is defined categorically, it is preserved by equivalences. Thus for example

$$\exists_f \|R\| \cong \|\exists f(R)\| \quad \text{and} \quad \forall_f \|R\| \cong \|\forall f(R)\|$$

by taking left and right adjoints in the above square.

The topos $C[P]$ looks "generated by 1" over C, in the following sense. Given a P-object $(X,=)$, by 2.5(iii) the equality predicate represents an epimorphism from $\|\underline{x} \in X\|$ to $(X,=)$. So we have

$$\begin{array}{ccc} \| \underline{x}\in X \| & \longrightarrow\!\!\!\!\!\rightarrow & (X,=) \\ \big\downarrow & & \\ \Delta_P X & & \end{array}$$

in $C[P]$. Of the possible representations of $(X,=)$ as a subquotient of a constant P-object, we can single out one with particularly good properties. Consider the predicate

$$S_X = [\![\exists \underline{x}(\underline{x}\in X \wedge \forall \underline{x}'(\underline{x}'\in_X \underline{R} \leftrightarrow \underline{x}' = \underline{x}))]\!]$$

in $P(PX)$ ("$\underline{R}$ is a singleton for $(X,=)$"), giving a canonical monomorphism $\| S_X \| \rightarrowtail \Delta_P(PX)$. Now $[\![S_X(\underline{R}) \wedge \underline{x}\in_X \underline{R}]\!] \in P(PX\times X)$ represents a map $b_X : \| S_X \| \longrightarrow \widetilde{(X,=)}$ in $C[P]$ (which by 2.5(iii) is an epimorphism). Classifying the partial map

$$\begin{array}{ccc} \| S_X \| & \xrightarrow{\ b_X\ }\!\!\!\!\rightarrow & (X,=) \\ \big\downarrow & & \\ \Delta_P(PX) & & \end{array}$$

gives a map $\beta_X : \Delta_P(PX) \longrightarrow \widetilde{(X,=)}$. By 2.6(iii), β_X is represented by $B_X = [\![\forall \underline{x}(S_X(\underline{R}) \wedge \underline{x}\in_X \underline{R} \leftrightarrow \underline{x}\in_X \underline{R}')]\!]$ in $P(PX\times PX)$. Note that since $P \models \underline{R}'\in X \wedge \underline{x}\in_X \underline{R}' \rightarrow S_X(\underline{R}')$, we have $P \models \underline{R}'\in X \rightarrow B_X(\underline{R}',\underline{R}')$ so that β_X is itself an epimorphism. It has the following property:

<u>3.7 Lemma</u>

Given any map $f : \Delta_P Y \longrightarrow \widetilde{(X,=)}$ in $C[P]$, there is a map $g : Y \longrightarrow PX$ in C such that

$$\begin{array}{ccc} \Delta_P(PX) & \xrightarrow{\ \beta_X\ }\!\!\!\!\rightarrow & \widetilde{(X,=)} \\ {\scriptstyle \Delta_P(g)}\big\uparrow & \nearrow{\scriptstyle f} & \\ \Delta_P(Y) & & \end{array}$$

commutes.

<u>Proof</u>

Pick a representative $F \in P(Y\times PX)$ for f. By (c) of Definiton 1.1, we can find $g : Y \longrightarrow PX$ in C such that

$$P \models \underline{x}\epsilon_X g(\underline{y}) \leftrightarrow \exists \underline{R}\ (F(\underline{y},\underline{R}) \wedge \underline{x}\epsilon_X \underline{R}).$$

Now (using Lemma 3.5) $\beta_X \circ \Delta_P(g)$ is represented by

$$[\![\forall \underline{x}(S_X(g\underline{y}) \wedge \underline{x}\epsilon_X g(\underline{y}) \leftrightarrow \underline{x}\epsilon_X \underline{R})]\!]$$

in $P(Y \times PX)$. But by the defining property of g and the fact that F is a functional relation from $(Y,=)$ to $(X,=)$, we have that

$$P \models F(\underline{y},\underline{R}) \wedge S_X(g\underline{y}) \wedge \underline{x}\epsilon_X g(\underline{y}) \rightarrow \underline{x}\epsilon_X \underline{R},$$

and $\quad P \models F(\underline{y},\underline{R}) \wedge \underline{x}\epsilon_X \underline{R} \rightarrow S_X(g\underline{y}) \wedge \underline{x}\epsilon_X g(\underline{y}),$

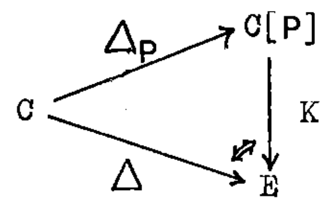
so that $F \vdash [\![\forall \underline{x}(S_X(g\underline{y}) \wedge \underline{x}\epsilon_X g(\underline{y}) \leftrightarrow \underline{x}\epsilon_X \underline{R})]\!]$ in $P(Y \times PX)$. It follows from the Remark after 2.2 that $f = \beta_X \circ \Delta_P(g)$. □

Thus in a weak sense (since the map g in Lemma 3.7 is not uniquely determined by f), β_X classifies the universal representation of $(X,=)$ as a subquotient of a constant P-object. We shall see that it is exactly this property of toposes equipped with a left exact functor from C that characterises the ones that arise from C-triposes.

Suppose then that $\Delta : C \longrightarrow E$ is a functor into a topos E that preserves finite limits. In view of Proposition 3.6, the obvious candidate for a C-tripos is $P = \mathbf{Sub}_E \circ \Delta^{op}$. Indeed, since Δ is left exact (and $\mathbf{Sub}_E$ is a tripos!) P satisfies parts (a) and (b) of Definition 1.1. Thus by the Remark after 2.3, we can constuct the category $C[P]$, and similarly we can still define the functor $\Delta_P : C \longrightarrow C[P]$, and it will be left exact.

3.8 Proposition

With E, Δ and P as above, there is a fully faithful comparison functor K from $C[P]$ to E such that



commutes up to isomorphism. K is an equivalence iff every object of E is the subquotient of some $\Delta(X)$ (X an object of C).

<u>Proof</u>

(i) Given a P-object (X,=), the equality $[\![\underline{x}=\underline{x}']\!] \rightarrowtail \Delta(X\times X)$ restricts to an equivalence relation on $[\![\underline{x}\in X]\!]$, with quotient K(X,=) say; thus

$$[\![\underline{x}=\underline{x}']\!] \rightrightarrows [\![\underline{x}\in X]\!] \xrightarrow{q_X} K(X,=)$$

is a coequalizer in E.

Given an arrow $f:(X,=) \longrightarrow (Y,=)$ in C[P], any represenative $F \rightarrowtail \Delta(X\times Y) \cong \Delta X\times\Delta Y$ factors through $[\![\underline{x}\in X]\!]\times[\![\underline{y}\in Y]\!] \rightarrowtail \Delta(X\times Y)$; take its image along $q_X\times q_Y$:

$$\begin{array}{ccc} F & \rightarrowtail & [\![\underline{x}\in X]\!] \times [\![\underline{y}\in Y]\!] \\ \downarrow & & \downarrow{\scriptstyle q_X\times q_Y} \\ K_F & \rightarrowtail & K(X,=)\times K(Y,=) \end{array}$$

Then the above square is also a pullback and K_F is the graph of a function $K(f):K(X,=) \longrightarrow K(Y,=)$ in E which is independent of the choice of representative F for f.

It is easy to see from the way that identities and composites are defined in C[P], that these definitions give a functor

$$K:C[P] \longrightarrow E.$$

(ii) Suppose that G is another functional relation from (X,=) to (Y,=), with $K_F \cong K_G$ in $Sub_E(K(X,=)\times K(Y,=))$. Then since the square above is a pullback, $F \cong G$ in $Sub_E\Delta(X\times Y)$. Thus K is faithful.

(iii) To see that K is full, given $\theta:K(X,=) \longrightarrow K(Y,=)$ in E, form the pullback

$$\begin{array}{ccccc} F & \rightarrowtail & [\![\underline{x}\in X]\!]\times[\![\underline{y}\in Y]\!] & \rightarrowtail & \Delta(X\times Y) \\ \downarrow & & \downarrow{\scriptstyle q_X\times q_Y} & & \\ K(X,=) & \underset{<id,\theta>}{\rightarrowtail} & K(X,=)\times K(Y,=) & & \end{array}$$

Then F represents a map $f:(X,=) \longrightarrow (Y,=)$ and by construction $K(f) = \theta$.

(iv) Given an object X of C, since Δ is left exact the constant equality on X, $[\![\underline{x} =_{\Delta X} \underline{x}']\!] \rightarrowtail \Delta(X\times X)$ is just the diagonal subobject of ΔX: hence $K(\Delta_P X) \cong \Delta X$, and this isomorphism is natural in X.

(v) Using choice, K will be an equivalence iff it is essentially surjective, i.e. iff for each object A of E there is some **P**-object (X,=) with $K(X,=) \cong A$. Given such an isomorphism, since $K(X,=)$ is a subquotient of ΔX, so is A. Conversely, given a diagram

$$\begin{array}{ccc} M & \xrightarrow{\ e\ } \!\!\!\rightarrow & A \\ \big\downarrow & & \\ \Delta X & & \end{array}$$

in E, the epimorphism e is the coequalizer of some equivalence relation $R \rightrightarrows M$. Then $R \rightarrowtail M\times M \rightarrowtail \Delta X \times \Delta X \cong \Delta(X\times X)$ is an equality predicate for X, and by its construction $K(X,R)$ is isomorphic to A. □

3.9 Example

Even if the functor K constructed in 3.8 is an equivalence, so that C[**P**] is a topos, **P** need not be a C-tripos. For example let E = Set[**P**] with **P** defined from a Heyting algebra H as in 1.8. We saw in 2.9 that if H is Boolean then E is a topos (and in fact Δ:Set $\longrightarrow$ E is logical). By the way E is constructed, the comparison functor K:Set[$\mathbf{Sub}_E \circ \Delta^{op}$] $\longrightarrow$ E is an equivalence. But $\mathbf{Sub}_E \circ \Delta^{op}$ is equivalent to **P** (as a Set-indexed category) and as we noted in 1.8, so long as H is infinite, part (c) of Definition 1.1 fails for **P**.

When $\mathbf{P} = \mathbf{Sub}_E \circ \Delta^{op}$ we may restate (c) of Definition 1.1 as:

(c″) For each object I of C there is an object PI of C and a map $\varepsilon_I : \Delta(PI) \longrightarrow (\Omega_E)^{\Delta I}$ in E such that

$$\begin{array}{ccc} C(J,PI) & \longrightarrow & E(\Delta J, (\Omega_E)^{\Delta I}) \\ f & \longmapsto & \varepsilon_I \circ \Delta f \end{array}$$

is a surjection, each object J of C.

Using this formulation, we can now prove the result that was promised after Lemma 3.7:

<u>3.10 Theorem</u>

Let C be a finitely complete category, E a topos and $\Delta : C \longrightarrow E$ a left exact functor. Then the following are equivalent:

(i) $\Delta : C \longrightarrow E$ is equivalent (over C) to $\Delta_{\mathbf{P}} : C \longrightarrow C[\mathbf{P}]$, for some C-tripos **P**;

(ii)(a) for each object A of E, there is some object $\hat{A}$ of C and a map $\beta_A : \Delta\hat{A} \longrightarrow \tilde{A}$ in E ($\tilde{A}$ the partial map classifier of A), such that given any $f : \Delta X \longrightarrow \tilde{A}$ in E there is $g : X \longrightarrow \hat{A}$ in C with $f = \beta_A \circ \Delta g$; and

(b) the maps β_A are all epimorphisms.

<u>Proof</u>

Since (ii) holds for $\Delta_{\mathbf{P}} : C \longrightarrow C[\mathbf{P}]$ by Lemma 3.7, we have that (i) implies (ii).

Conversely suppose (ii) holds. By (ii)(b), the partial map that β_A classifies presents A as a subquotient of $\Delta(\hat{A})$:

$$\begin{array}{ccc} \cdot & \twoheadrightarrow & A \\ \downarrow & \text{pb} & \downarrow \eta_A \\ \Delta(\tilde{A}) & \underset{\beta_A}{\twoheadrightarrow} & \tilde{A} \end{array}$$

So putting $\mathbf{P} = \mathbf{Sub}_E \circ \Delta^{op}$ we have by Proposition 3.8 that $\Delta : C \longrightarrow E$ is equivalent (over C) to $\Delta_P : C \longrightarrow C[\mathbf{P}]$. By the remarks after that proposition, to prove that $\mathbf{P}$ is actually a C-tripos, it suffices to verify condition (c").

But given an object I of C, we may take $PI = ((\Omega_E)^{\Delta I})^{\wedge}$; since there is a retraction

$$\Omega^{\Delta I} \rightarrowtail^{\eta} \widetilde{(\Omega^{\Delta I})} \xrightarrow{r} \twoheadrightarrow \Omega^{\Delta I} \quad ; \quad r\circ\eta = \mathrm{id}$$

(where r is $\widetilde{(\Omega^{\Delta I})} \rightarrowtail \Omega^{\Omega^{\Delta I}} \xrightarrow{\bigcap} \Omega^{\Delta I}$), and by (ii)(a), we have that $\varepsilon_I = r\circ\beta_{\Omega^{\Delta I}}$ induces a surjection

$$\begin{array}{ccccc} C(J,PI) & \twoheadrightarrow & E(\Delta J, \widetilde{\Omega^{\Delta I}}) & \twoheadrightarrow & E(\Delta J, \Omega^{\Delta I}) \\ g & \longmapsto & \beta\circ\Delta g & \longmapsto & (r\circ\beta)\circ\Delta g, \end{array}$$

each object J of C.

□

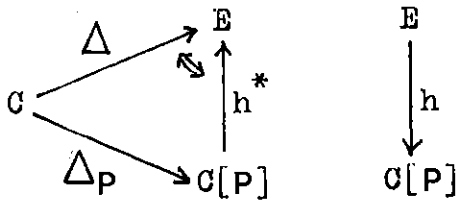
Note that condition (c") above is satisfied if Δ has a right adjoint Γ. For then we may take $PI = \Gamma(\Omega_E)^{\Delta I}$ and $\varepsilon_I : \Delta PI \longrightarrow (\Omega_E)^{\Delta I}$ to be the counit of $\Delta \dashv \Gamma$ at $(\Omega_E)^{\Delta I}$; then

$$\begin{array}{ccc} C(J,PI) & \longrightarrow & E(\Delta J, (\Omega_E)^{\Delta I}) \\ f & \longmapsto & \varepsilon_I \circ \Delta f \end{array}$$

is actually a bijection. In particular, this shows that given a geometric morphism of toposes $f : F \longrightarrow E$, $\mathbf{Sub}_F \circ (f^*)^{op}$ is an E-tripos. It is not hard to see that $E[\mathbf{Sub}_F \circ (f^*)^{op}] \simeq E[f_*\Omega_F]$, E-valued sheaves on the internal locale $f_*\Omega_F$; the full and faithful comparison functor $K : E[f_*\Omega_F] \longrightarrow F$ of Proposition 3.8 is in this case the inverse image part of the geometric morphism $F \longrightarrow E[f_*\Omega_F]$ which forms half the "hyperconnected-localic" factorization of f (see [11]). When dealing with triposes in general we lose the localic part, but retain the hyperconnected part:

3.11 Proposition

With $\Delta : C \longrightarrow E$ as in Theorem 3.10, suppose Δ satisfies condition (ii)(a) of that theorem. Then $P = \mathbf{Sub}_E \circ \Delta^{op}$ is a C-tripos and there is a hyperconnected geometric morphism $h: E \longrightarrow C[P]$ such that $h^* \circ \Delta_P \cong \Delta$.



Proof

That P is a C-tripos follows as in the proof of Theorem 3.10, and by Proposition 3.8 we have a fully faithful comparison functor $K: C[P] \longrightarrow E$ with $K \circ \Delta_P \cong \Delta$.

Now it is not hard to see that K is left exact and that its image in E is closed under taking subobjects there. So if we can show that K has a right adjoint we will have, by 1.5 of [11], that it is the inverse image part of a hyperconnected geometric morphism $E \longrightarrow C[P]$.

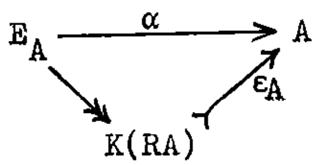
But given A in E we have $\beta_A: \Delta(\hat{A}) \longrightarrow \tilde{A}$, classifying the partial map

$$\begin{array}{ccc} E_A & \xrightarrow{\alpha} & A \\ \downarrow & & \\ \Delta(\hat{A}) & & \end{array}$$

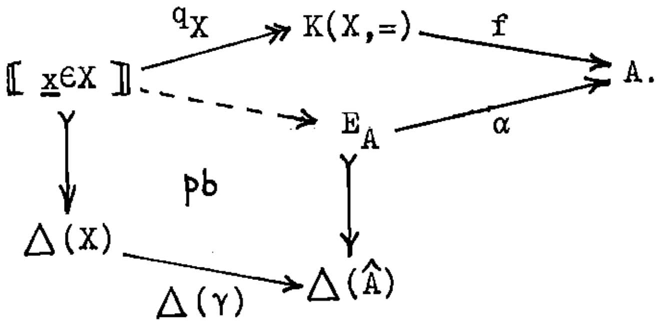
say. Then the kernel pair of α, $kpr(\alpha) \rightrightarrows E_A$, gives an equality predicate

$$kpr(\alpha) \rightarrowtail E_A \times E_A \rightarrowtail \Delta\hat{A} \times \Delta\hat{A} = \Delta(\hat{A} \times \hat{A})$$

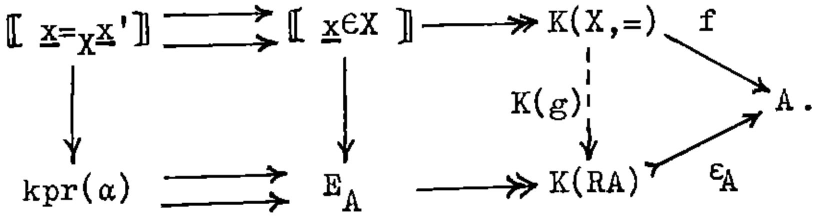
on $\hat{A}$, making it into a P-object $(\hat{A}, kpr(\alpha)) = R(A)$. Furthermore $K(RA)$ is obtained by coequalizing $kpr(\alpha) \rightrightarrows E_A$, so we get



Then ε_A is universal amongst maps of the form $K(X,=) \xrightarrow{f} A$. For given such an f, by (ii)(a) there is $\gamma: X \longrightarrow \hat{A}$ in C with



Then pulling back the graph of $\Delta(\gamma)$ along $[\![\, \underline{x} \in X \,]\!] \times E_A \rightarrowtail \Delta X \times \Delta A$ gives a predicate $G \rightarrowtail \Delta(X \times A)$ which represents a map $g:(X,=) \longrightarrow RA$ in $C[P]$, and $K(g)$ is the factorization of f through ε_A:



Since ε_A is mono and K faithful, g is the unique such map.

Thus ε_A is the counit at A of an adjunction $K \dashv R$, as required.

□

3.12 Change of base

Suppose that $f:F \longrightarrow E$ is a geometric morphism of toposes. Using Proposition 3.11, we can generalise the process which sends an F-locale A along f_* to an E-locale f_*A, with the resulting (hyperconnected) geometric morphism $F[A] \longrightarrow E[f_*A]$.

For let $P = F(-,\Sigma)$ be a canonically presented F-tripos. Since $\Delta_P : F \longrightarrow F[P]$ satisfies (ii)(a) of 3.10 and f^* has a right adjoint, $\Delta_P \circ f^*$ will also satisfy it (for a P-object $(Y,=)$, the map $\Delta_P f^*(f_*\Sigma^Y) \xrightarrow{\Delta_P(\varepsilon_{\Sigma^Y})} \Delta_P(\Sigma^Y) \xrightarrow{\beta_Y} (Y,=)$ will do the trick). So $\mathrm{Sub}_{F[P]} \circ (\Delta_P f^*)^{op}$ is an E-tripos: in view of the natural isomorphisms

$$\mathrm{Sub}_{F[P]}(\Delta_P f^* X) \cong P(f^* X) = F(f^* X, \Sigma) \cong E(X, f_*\Sigma)$$

we can replace it with a canonically presented tripos structure on $E(-,f_*\Sigma)$. We shall write f_*P for this tripos.

Thus we have a hyperconnected geometric morphism $h:F[P] \longrightarrow E[f_*P]$ such that

$$\begin{array}{ccc} F[P] & \xleftarrow{\;h^*\;} & E[f_*P] \\ {\scriptstyle \Delta_P}\uparrow & \cong & \uparrow{\scriptstyle \Delta_{f_*P}} \\ F & \xleftarrow{\;f^*\;} & E \end{array}$$

commutes up to isomorphism.

In general $\Delta_P : C \longrightarrow C[P]$ will not have a right adjoint (for example, when P is a realizability tripos), but it may still have some "right exactness". In fact, for all the examples of triposes defined over toposes given in Chapter 1, Δ_P preserves epimorphisms. As we shall now see, this is due to the way in which quantification is defined in them:

<u>3.13 Lemma</u>

Let C be a finitely complete category and P a C-tripos. Given $f:X \longrightarrow Y$ in C,

$$id_{PY} \dashv\vdash \forall f \circ Pf \quad \text{iff} \quad id_{PY} \dashv\vdash \exists f \circ Pf \quad \text{iff} \quad \top_Y \vdash (\exists f)\top_X .$$

The functor Δ_P preserves epimorphisms iff these conditions are satisfied for all epimorphisms f in C.

<u>Proof</u>

By category theory, $id_{PY} \dashv\vdash \forall f \circ Pf$ iff Pf is a fully faithful functor, iff $id_{PY} \dashv\vdash \exists f \circ Pf$, which trivially implies that $\top_Y \vdash (\exists f)\top_X$. But if the latter holds, then given ψ in P(Y)

$$\psi \dashv\vdash \psi \wedge (\exists f)\top_X \dashv\vdash \exists f(Pf\,\psi \wedge \top_X) \dashv\vdash (\exists f)(Pf)\psi .$$

Now since $[\![\, \underline{y} \in \Delta_P Y \,]\!] = \top_Y$ and $\exists\langle id_X, f\rangle \top_X \in P(X \times Y)$ represents $\Delta_P(f)$, we have from 2.5(iii) that $\Delta_P(f)$ is epimorphic iff $\top_Y \vdash (\exists \pi_2)\exists\langle id_X, f\rangle \top_X \dashv\vdash (\exists f)\top_X$. So Δ_P preserves epimorphisms iff $\top_Y \vdash (\exists f)\top_X$, for all epi's f.

$\square$

Recall that if P is an E-tripos, for E a topos, we may assume up to equivalence that it is canonically presented, say $P = E(-,\Sigma)$. In 1.2 we said that P had fibre-wise quantification iff there were maps $\bigvee_P, \bigwedge_P : (\Omega_E)^\Sigma \longrightarrow \Sigma$ with

$$(\exists f\varphi)\underline{y} = \bigvee_P \{ \varphi\underline{x} \mid f\underline{x}=\underline{y} \} \quad \text{and} \quad (\forall f\varphi)\underline{y} = \bigwedge_P \{ \varphi\underline{x} \mid f\underline{x}=\underline{y} \},$$

given maps $f: X \longrightarrow Y$ and $\varphi: X \longrightarrow \Sigma$ in E. Note that if this is the case, the maps $\bigwedge_P, \bigvee_P$ are uniquely determined by $\forall$ and $\exists$. For let $\langle e,n\rangle : \in_\Sigma \rightarrowtail \Sigma \times (\Omega_E)^\Sigma$ be the (standard) membership relation for Σ in E. By assumption, $(\forall n)e = \bigwedge_P \circ s$ and $(\exists n)e = \bigvee_P \circ s$, where $s: \Omega^\Sigma \longrightarrow \Omega^\Sigma$ sends $\underline{p}$ to $\{ e(\underline{\varepsilon}) \mid n(\underline{\varepsilon})=\underline{p} \}$. Thus $s = id_{\Omega^\Sigma}$, and so $\bigwedge_P = (\forall n)e$, $\bigvee_P = (\exists n)e$.

3.14 Proposition

Let E be a topos and P an E-tripos. Then we can choose $\forall$ and $\exists$ so that P has fibre-wise quantification iff $\Delta_P : E \longrightarrow E[P]$ preserves epimorphisms.

Proof

From the above remarks, we have to check that the maps $\bigwedge_P = (\forall n)e$ and $\bigvee_P = (\exists n)e$ work.

Given $f: X \longrightarrow Y$ and $\varphi: X \longrightarrow \Sigma$ in E, form the image factorisation of $\langle\varphi, f\rangle$:

$$\begin{array}{ccc} X & \xrightarrow{\langle\varphi,f\rangle} & \Sigma \times Y. \\ & {\scriptstyle q}\searrow \quad \nearrow{\scriptstyle \langle\varphi',f'\rangle} & \\ & K & \end{array}$$

Then there is a pullback square

$$\begin{array}{ccc} K & \xrightarrow{\ r\ } & \in_\Sigma \\ {\scriptstyle f'}\downarrow & pb & \downarrow{\scriptstyle n} \\ Y & \xrightarrow{\ s\ } & (\Omega_E)^\Sigma \end{array}$$

where $s\underline{y} = \{ \varphi\underline{x} \mid f\underline{x}=\underline{y} \}$ and $e \circ r = \varphi'$. Then by the Beck conditions

$$\left.\begin{array}{l} \bigwedge_P s = (Ps)(\forall n)e \dashv\vdash (\forall f')(Pr)e = (\forall f')\varphi' \\ \text{and } \bigvee_P s = (Ps)(\exists n)e \dashv\vdash (\exists f')(Pr)e = (\exists f')\varphi' \end{array}\right\}(*).$$

Now suppose that Δ_P preserves epimorphisms. Then by Lemma 3.12

$$\forall q \circ Pq \dashv\vdash id_{PX} \dashv\vdash \exists q \circ Pq,$$

so from (*)

$$\bigwedge_P \circ s \dashv\vdash (\forall f')(\forall q)(Pq)\varphi' \dashv\vdash (\forall f)\varphi,$$

$$\text{and } \bigvee_P \circ s \dashv\vdash (\exists f')(\exists q)(Pq)\varphi' \dashv\vdash (\exists f)\varphi.$$

So we may redefine the quantifiers in terms of $\bigwedge_P, \bigvee_P$ as required.

Conversely if P has fibre-wise quantification, given an epimorphism $f:X \twoheadrightarrow Y$, putting $\varphi = \top_X$ in the above, we can take $K = Y$, $q = f$, $f' = id_Y$ and $\varphi' = \top_Y$. So from (*)

$$(\exists f)\top_X = (\exists f)\varphi = \bigvee_P \circ s \dashv\vdash (\exists f')\varphi' \dashv\vdash \top_Y.$$

Thus by Lemma 3.13, Δ_P preserves epimorphisms.

□

As a corollary of Proposition 3.14, assuming the Axiom of Choice, we can choose qantification fibre-wise in any Set-tripos.

4. MORPHISMS OVER A FIXED BASE

In this chapter we wish to examine the relationship between C-indexed functors $\mathbf{P} \longrightarrow \mathbf{R}$ between C-triposes (C a fixed category with finite limits) and functors between the corresponding toposes $C[\mathbf{P}] \longrightarrow C[\mathbf{R}]$. Once again we have in mind the localic case, where continuous maps between E-locales (for E a topos) correspond to geometric morphisms between their associated sheaf toposes. We shall see that, with the correct definitions, the same remains true of triposes.

Suppose that $l:\mathbf{P} \longrightarrow \mathbf{R}$ is a C-indexed functor between C-triposes. Given a $\mathbf{P}$-object $(X,=)$, $l(=) \in \mathbf{R}(X \times X)$ will be an equality predicate on X provided each $l_X:\mathbf{P}X \longrightarrow \mathbf{R}X$ preserves finite meets, i.e. if l is a left-exact C-indexed functor. Supposing this to be the case, we get an $\mathbf{R}$-object $(X,l(=))$, which we shall denote $\bar{l}(X,=)$.

However, if l preserves $\top$ and $\wedge$ but not necessarily $\exists$, then given a functional relation $F \in \mathbf{P}(X \times Y)$ between $\mathbf{P}$-objects $(X,=)$ and $(Y,=)$, $l(F) \in \mathbf{R}(X \times Y)$ will only be a partial functional relation (c.f. 2.6(iii)). So how can we extend $\bar{l}$ to maps in $C[\mathbf{P}]$?

In the localic case we would use the fact that each H-set was isomorphic to the underlying H-set of a sheaf (namely the sheaf generated by the H-set: c.f. 4.17 of [6]) and define $\bar{l}$ on sheaves. In the general case we can still recover enough of this to do the trick.

Given a $\mathbf{P}$-object $(X,=)$, recall from Chapter 3 that $S_X \in \mathbf{P}(\mathrm{P}X)$ is the predicate

$$[\![\exists \underline{x}(\underline{x} \in X \wedge \forall \underline{x}'(\underline{x}' \in_X \underline{R} \leftrightarrow \underline{x}' = \underline{x}))]\!].$$

Now it is easy to see that S_X is a strict relation for $\mathrm{P}(X,=)$, the powerobject of (X,=): so it determines a canonical monic, which we shall denote by

$$S(X,=) \rightarrowtail P(X,=).$$

The diagram

$$\begin{array}{ccc} \| S_X \| & \xrightarrow{b_X} & (X,=) \\ \downarrow & & \\ \Delta_P(PX) & & \end{array}$$

constructed for Lemma 3.7 then gives us an isomorphism

$$(X,=) \xrightarrow{\cong} S(X,=),$$

represented by $[\![\underline{x} \in_X \underline{R} \wedge S_X(\underline{R})]\!]$ in $P(X \times PX)$. (In the localic case S(X,=) would be the underlying H-set of the sheaf generated by (X,=): see 4.18 of [6]).

4.1 Definition

Say that a P-object (X,=) is weakly complete iff given any partial functional relation F from (Y,=) to (X,=), there is $f:Y \longrightarrow X$ in C with

$$P \models \exists \underline{x}\, F(\underline{y},\underline{x}) \leftrightarrow F(\underline{y},f\underline{y}).$$

4.2 Proposition

Any P-object is isomorphic to a weakly complete one.

Proof

From the above, it will suffice to show that, given a P-object (X,=), S(X,=) is weakly complete. But this follows from Lemma 3.7 (or directly as in Proposition 3.3 of [9]).

□

4.3 Lemma

Suppose that $l:P \longrightarrow R$ is a left exact C-indexed functor between C-triposes, and that (X,=),(Y,=) are P-objects, the latter being weakly complete. Then given any functional relation F from (X,=) to (Y,=), l(F) is a functional relation from $\bar{l}(X,=)$ to $\bar{l}(Y,=)$.

Proof

From previous remarks, all we need verify is that $l(F)$ is total. But since $(Y,=)$ is weakly complete, we can find $f:X \longrightarrow Y$ with $[\![\exists \underline{y} F(\underline{x},\underline{y})]\!] \dashv\vdash [\![F(\underline{x},f\underline{x})]\!]$. Then we have

$$\begin{aligned} l[\![\underline{x} \in X]\!]_P &\vdash l[\![\exists \underline{y} F(\underline{x},\underline{y})]\!]_P \quad \text{(since } F \text{ is total)} \\ &\dashv\vdash l[\![F(\underline{x},f\underline{x})]\!]_P \\ &\dashv\vdash [\![(lF)(\underline{x},f\underline{x})]\!]_R \\ &\dashv\vdash [\![\exists \underline{y} (lF)(\underline{x},\underline{y})]\!]_R \end{aligned}$$

as required.

□

Thus given $f:(X,=) \longrightarrow (Y,=)$ in $C[P]$, represented by F, with $(Y,=)$ weakly complete, we can define $\bar{l}(f):\bar{l}(X,=) \longrightarrow \bar{l}(Y,=)$ to be represented by $l(F)$. This makes $\bar{l}$ into a left exact functor on the full subcategory of weakly complete P-objects (since, as in the proof of Lemma 4.3, l preserves the existential quantifiers used in constructing composites of maps and finite limits). Then, using Proposition 4.2, we can extend $\bar{l}$ to a left exact functor $C[P] \longrightarrow C[R]$.

4.4 Remark

Since $l_I:P(I) \longrightarrow R(I)$ is natural in I, it follows from part (c) of Definition 1.1 that l is determined up to isomorphism by what it does to the predicate $\in_1$ in $P(1 \times P1)$. In particular if $\mathbb{E}$ is a topos, $P = \mathbb{E}(-,\Sigma)$, $R = \mathbb{E}(-,\Lambda)$ are canonically presented $\mathbb{E}$-triposes and $l:P \longrightarrow R$ is an $\mathbb{E}$-indexed functor, then l is determined up to isomorphism by $l_\Sigma(id_\Sigma):\Sigma \longrightarrow \Lambda$: given any $\varphi:I \longrightarrow \Sigma$ we have

$$l_I(\varphi) = l_I(P\varphi)id_\Sigma \dashv\vdash_I (R\varphi)l_\Sigma(id_\Sigma) = (l_\Sigma id_\Sigma)\circ\varphi.$$

We shall usually just write l for $l_\Sigma(id_\Sigma)$, and assume $l_I(\varphi) = l\circ\varphi$.

4.5 Example

Let E be a topos and $\mathbf{P} = E(-,\Sigma)$ a canonically presented E-tripos. There is an E-indexed functor $\delta:\mathbf{Sub}_E \longrightarrow \mathbf{P}$, which if we make $\mathbf{Sub}_E$ canonically presented by replacing it with $E(-,\Omega_E)$, is given by $\delta = (\exists \mathrm{true}_E)\top_1 : \Omega_E \longrightarrow \Sigma$ (where $\mathrm{true}_E : 1 \longrightarrow \Omega_E$ and $\top_1 : 1 \longrightarrow \Sigma$ are the respective top elements). Thus if $\varphi : I \longrightarrow \Omega_E$ classifies $\alpha : A \rightarrowtail I$, by the Beck condition for $\exists$, we have

$$\delta\varphi = (\mathbf{P}\varphi)(\exists \mathrm{true}_E)\top_1 \dashv\vdash_I (\exists\alpha)(\mathbf{P}A)\top_1 = (\exists\alpha)\top_A.$$

From this it easily follows that δ is left exact. Furthermore, identifying $E[\mathbf{Sub}_E]$ with E, we find that $\bar{\delta}:E[\mathbf{Sub}_E] \longrightarrow E[\mathbf{P}]$ is actually the functor $\Delta_{\mathbf{P}}:E \longrightarrow E[\mathbf{P}]$ constructed in Chapter 3.

We now turn to the analogue of continuous maps between locales:

4.6 Definition

Let C be a finitely complete category and $\mathbf{P},\mathbf{R}$ C-triposes. A geometric morphism $f:\mathbf{P} \longrightarrow \mathbf{R}$ is given by a pair of C-indexed functors $f^*:\mathbf{R} \longrightarrow \mathbf{P}$, $f_*:\mathbf{P} \longrightarrow \mathbf{R}$, such that for each object X of C, $(f^*)_X$ is left exact and left adjoint to $(f_*)_X$.

Given such a geometric morphism f, since f^*, f_* are left exact we get induced left exact functors $\bar{f}^*, \bar{f}_*$ between $C[\mathbf{P}]$ and $C[\mathbf{R}]$ defined as above. But note that since f_* commutes with substitution along maps (up to isomorphism), on taking left adjoints we get that f^* preserves $\exists$. It follows that the construction of $\bar{f}^*$ on the full subcategory of weakly complete $\mathbf{P}$-objects works on the whole of $C[\mathbf{R}]$: so we can construct $\bar{f}^*$ without recourse to weak completions.

4.7 Proposition

Let $f\colon \mathsf{P} \longrightarrow \mathsf{R}$ be a geometric morphism of C-triposes. Then $\bar{f}^*$ and $\bar{f}_*$ constitute a geometric morphism of toposes $\bar{f}\colon C[\mathsf{P}] \longrightarrow C[\mathsf{R}]$.

Proof

It suffices to check that $\bar{f}^*$ is left adjoint to $\bar{f}_*$. To do this, we will exhibit a map $\varepsilon_{(X,=)}\colon \bar{f}^*\bar{f}_*(X,=) \longrightarrow (X,=)$ in $C[\mathsf{P}]$ which is universal amongst maps $\bar{f}^*(Y,=) \longrightarrow (X,=)$ ($(Y,=)$ an R-object).

By Proposition 4.2, we may assume that $(X,=)$ is weakly complete. Consider $E = f^*f_*[\![\, \underline{x} \in X \,]\!] \wedge [\![\, \underline{x}=\underline{x}']\!]$ in $\mathsf{P}(X\times X)$. As $f^*f_* \vdash \mathrm{id}$, E is a functional relation from $\bar{f}^*\bar{f}_*(X,=)$ to $(X,=)$, and so represents a map $\varepsilon_{(X,=)}\colon \bar{f}^*\bar{f}_*(X,=) \longrightarrow (X,=)$.

Given any $g\colon \bar{f}^*(Y,=) \longrightarrow (X,=)$, represented by $G \in \mathsf{P}(Y\times X)$ say, consider $\bar{G} = f_*G \wedge [\![\, \underline{y} \in Y \,]\!]$. Since $\mathrm{id} \vdash f_*f^*$, $\bar{G}$ is a strict, single-valued relation on $(Y,=)\times\bar{f}_*(X,=)$. Moreover since $(X,=)$ is weakly complete, as in Lemma 4.3 we have

$$f_*(f^*[\![\, \underline{y} \in Y \,]\!]) \vdash [\![\, \exists \underline{x}\, (f_*G)(\underline{y},\underline{x}) \,]\!].$$

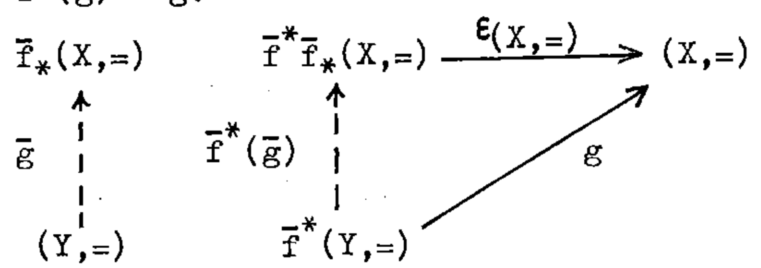
Then since $\mathrm{id} \vdash f_*f^*$ we have

$$[\![\, \underline{y} \in Y \,]\!] \vdash [\![\, \exists \underline{x}\, \bar{G}(\underline{y},\underline{x}) \,]\!].$$

So $\bar{G}$ represents a map $\bar{g}\colon (Y,=) \longrightarrow \bar{f}_*(X,=)$ in $C[\mathsf{R}]$. Now $\varepsilon_{(X,=)} \circ \bar{f}^*(\bar{g})$ is represented by

$$[\![\, \exists \underline{x}' \big(f^*(f_*G(\underline{y},\underline{x}') \wedge \underline{y} \in Y) \wedge f^*f_*(\underline{x}' \in X) \wedge \underline{x}'=\underline{x}\big) \,]\!],$$

which entails G in $\mathsf{P}(Y\times X)$; so by the Remark after 2.2, we have that $\varepsilon_{(X,=)} \circ \bar{f}^*(\bar{g}) = g$.



It remains to show that $\bar{g}$ is the unique such map. Suppose $h:(Y,=) \longrightarrow \bar{f}^*(X,=)$ satisfies $\varepsilon_{(X,=)} \circ \bar{f}^*(h) = g$, and that h is represented by $H \in \mathbf{R}(Y \times X)$. Then

$$\mathbf{P} \models \exists \underline{x}' \, (f^*H(\underline{y},\underline{x}') \wedge f^*f_*(\underline{x}' \in X) \wedge \underline{x}'=\underline{x}) \;\leftrightarrow\; G(\underline{y},\underline{x}).$$

Thus $f^*H \vdash G$ in $\mathbf{P}(Y \times X)$, and hence

$$H \;\vdash\; f_*G \wedge [\![\, \underline{y} \in Y \,]\!] \;=\; \bar{G},$$

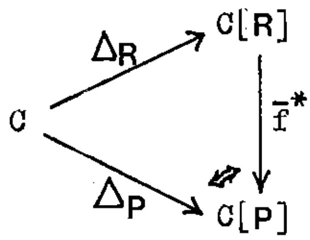
since H is strict. Therefore $h = \bar{g}$, as required.

□

Note that since f^* preserves $\top$ and $\exists$, we have that

$$f^*(=_{\Delta_R X}) \;\dashv\vdash\; =_{\Delta_P X}$$

in $\mathbf{P}(X \times X)$. Thus



commutes up to isomorphism, i.e. the inverse image part of the geometric morphism $\bar{f}: C[\mathbf{P}] \longrightarrow C[\mathbf{R}]$ constructed above preserves constant objects. Conversely, we have:

<u>4.8 Theorem</u>

Let $\mathbf{P},\mathbf{R}$ be C-triposes and $g: C[\mathbf{P}] \longrightarrow C[\mathbf{R}]$ a geometric morphism of toposes whose inverse image functor preserves constant objects (up to isomorphism). Then there is a geometric morphism of triposes $f: \mathbf{P} \longrightarrow \mathbf{R}$ unique up to (unique) isomorphism such that g is isomorphic to $\bar{f}$.

<u>Proof</u>

To define f use the equivalences $\mathbf{P} \simeq \mathbf{Sub}_{C[\mathbf{P}]} \circ \Delta_{\mathbf{P}}^{op}$, $\mathbf{R} \simeq \mathbf{Sub}_{C[\mathbf{R}]} \circ \Delta_{\mathbf{R}}^{op}$ of Proposition 3.6. Given S in $\mathbf{R}(X)$, define f^*S in $\mathbf{P}(X)$ up to isomorphism by requiring that

$$\| f^*S \| \rightarrowtail \Delta_P X \;\cong\; g^* \| S \| \rightarrowtail g^* \Delta_R X \;\cong\; \Delta_P X$$

in $\mathbf{Sub}_{C[\mathbf{P}]}(\Delta_{\mathbf{P}}X)$. This makes f^* into a left exact C-indexed functor $\mathbf{R} \longrightarrow \mathbf{P}$.

Similarly given R in $\mathbf{P}(X)$, define f_*R in $\mathbf{R}(X)$ up to isomorphism by requiring that

$$\begin{array}{ccc} \| f_*R \| & \rightarrowtail & \Delta_{\mathbf{R}}X \\ \downarrow & & \downarrow \eta_{\Delta_{\mathbf{R}}X} \\ g_*\|R\| & \rightarrowtail & g_*\Delta_{\mathbf{P}}X = g_*g^*\Delta_{\mathbf{R}}X \end{array}$$

be a pullback in $C[\mathbf{R}]$, where η is the unit of $g^* \dashv g_*$. Again, f_* is a C-indexed functor. Furthermore

$$\begin{array}{ll} & \mathbf{S} \vdash f_*R \quad \text{in } \mathbf{R}(X) \\ \text{iff} & \|S\| \leqslant \| f_*R \| \quad \text{in } \mathrm{Sub}(\Delta_{\mathbf{R}}X) \\ \text{iff} & (g^*\|S\| \rightarrowtail g^*\Delta_{\mathbf{R}}X \cong \Delta_{\mathbf{P}}X) \leqslant \|R\| \text{ in } \mathrm{Sub}(\Delta_{\mathbf{P}}X) \\ \text{iff} & f^*S \vdash R \quad \text{in } \mathbf{P}(X). \end{array}$$

So we have a geometric morphism of triposes $f: \mathbf{P} \longrightarrow \mathbf{R}$.

To show that $\bar{f}^*$ is isomorphic to g^*, from Proposition 3.8 we have that an $\mathbf{R}$-object $(X,=)$ is presented as the coequalizer of an equivalence relation

$$\| \underline{x}=\underline{x}' \| \rightrightarrows \| \underline{x}\in X \| \twoheadrightarrow (X,=),$$

and a map $r:(X,=) \longrightarrow (Y,=)$ (represented by R in $\mathbf{R}(X\times Y)$ say) is presented as a pullback/image factorisation

$$\begin{array}{ccc} \| R(\underline{x},\underline{y}) \| & \rightarrowtail & \| \underline{x}\in X \| \times \| \underline{y}\in Y \| \\ \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\ (X,=) & \xrightarrow[\langle id,r\rangle]{} & (X,=) \times (Y,=). \end{array}$$

Since g^* preserves the exactness of these diagrams and transforms the subobjects $\|\cdot\|$ into $\| f^*(\cdot) \|$ of the appropriate predicates, we get $g^*(X,=) \cong (X,f^*(=)) \cong \bar{f}^*(X,=)$, naturally in $(X,=)$.

Finally $\bar{f}$ is unique up to isomorphism, since if $g = \bar{f}'$, some geometric morphism $f': \mathbf{P} \longrightarrow \mathbf{R}$, then for S in $\mathbf{R}(X)$

$$\begin{array}{ccccc}
g^*\|S\| & \rightarrowtail & g^*\Delta_R X & & \\
\cong\downarrow & & \cong\downarrow & \searrow\cong & \\
\bar{f}'^*\|S\| & \rightarrowtail & \bar{f}'^*\Delta_R X & \xrightarrow{\cong} & \Delta_P X \\
\cong\downarrow & & & \nearrow & \\
\| f'^* S \| & \rightarrowtail & & &
\end{array}$$

commutes, so $f'^*S \dashv\vdash f^*S$. □

4.9 Examples of geometric morphisms

(i) Geometric morphisms between the canonical triposes of locales of course correspond to continuous maps between the locales themselves.

(ii) Let $\mathbb{A}$ be a combinatory algebra in a topos E, and let $\gamma : PA \longrightarrow \Omega_E$ classify the subobject $\{ \underline{p} \mid \exists \underline{a}(\underline{a} \in \underline{p}) \} \rightarrowtail PA$. Recalling the description of $\top_{\mathbb{A}}$ and $\wedge_{\mathbb{A}}$ from 1.5, we have

$$\gamma(\top_{\mathbb{A}}) = \mathrm{true}_E,$$

and $$E \models \forall \underline{p}, \underline{q} \in PA(\gamma\underline{p} \wedge \gamma\underline{q} \rightarrow \gamma(\underline{p} \wedge \underline{q})).$$

Similarly, if for $\varphi, \psi : I \longrightarrow PA$ we have

$$E \models (\ \forall I(\varphi \rightarrow_A \psi) \text{ inhabited}),$$

then $$E \models \forall \underline{i} \in I(\gamma\varphi\underline{i} \rightarrow \gamma\psi\underline{i}\,).$$

It follows that if P is a realizability tripos on $\mathbb{A}$ with designated truth-values given in either of the ways (i) or (ii) of 1.5, γ gives a left exact E-indexed functor $\mathbf{P} \longrightarrow \mathbf{Sub}_E$. Furthermore γ is left adjoint to the functor δ of Example 4.5 (since $\gamma \circ \delta = \mathrm{id}$ and $\lambda x.x = SKK \in PA(\mathrm{id} \rightarrow_{\mathbb{A}} \delta\gamma) \in P(1)$).

Thus γ and δ constitute a geometric morphism $\mathbf{Sub}_E \longrightarrow \mathbf{P}$ which we shall call η (for reasons that will become apparent in Chapter 7). By Proposition 4.7, η induces a geometric morphism $\bar{\eta} : E \longrightarrow E[\mathbf{P}]$, and we noted in 4.5 that $\bar{\eta}_*$ is just Δ_P. Similarly $\eta^* = \bar{\gamma} : E[\mathbf{P}] \longrightarrow E$ is just an internal global section functor and may be described as follows:

Given a P-object $(X,=)$, $\bar{\gamma}(X,=)$ is the quotient in E of $\{ \underline{x} \mid [\![\underline{x} \in X]\!] \text{ inhabited} \}$ by the equivalence relation $\{ (\underline{x},\underline{x}') \mid [\![\underline{x}=\underline{x}']\!] \text{ inhabited} \}$. And if $f:(X,=) \longrightarrow (Y,=)$ in $E[P]$, then $\bar{\gamma}f:\bar{\gamma}(X,=) \longrightarrow \bar{\gamma}(Y,=)$ has graph $\{ ([\underline{x}],[\underline{y}]) \mid F(\underline{x},\underline{y}) \text{ inhabited} \}$ (where F is any representative for f and $[\underline{x}]$ is the equivalence class of $\underline{x}$).

Thus the situation for realizability toposes is in some sense opposite to that for localic toposes, where taking global sections is <u>right</u> adjoint to the constant-sheaf functor.

<u>(iii)</u> Let **P** be the effective tripos (defined in 1.5) and **R** the tripos described in 1.6; thus $\mathbf{P} = \mathrm{Set}(-,P\mathbb{N})$, $\mathbf{R} = \mathrm{Set}(-,\Sigma)$ where $\Sigma = \{ R \subseteq \mathbb{N}\times\mathbb{N} \mid R \text{ is symmetric and transitive} \}$.

Consider the maps $i,d:P\mathbb{N} \longrightarrow \Sigma$, $u:\Sigma \longrightarrow P\mathbb{N}$ given by

$i(p) = p\times p$ (the indiscrete equivalence relation on p)

$d(p) = (p\times p)\cap \mathrm{im}\Delta$ (the discrete equivalence relation on p)

and $u(R) = \Delta^{-1}(R)$ (the underlying set of the partial equivalence relation R)

(where $\Delta:\mathbb{N} \longrightarrow \mathbb{N}\times\mathbb{N}$ is the diagonal map).

It is easily verified that u,i constitute a geometric morphism $i:\mathbf{P} \longrightarrow \mathbf{R}$, whilst d,u constitute a geometric morphism $r:\mathbf{R} \longrightarrow \mathbf{P}$; moreover $i^*r^* = u\circ d = \mathrm{id}_{P\mathbb{N}}$. Thus i,r induce geometric morphisms of toposes

$$\begin{array}{ccc} \mathrm{Eff} = \mathrm{Set}[\mathbf{P}] & \xrightarrow{\ \bar{i}\ } & \mathrm{Set}[\mathbf{R}] \\ & \searrow^{\mathrm{id}} & \downarrow \bar{r} \\ & & \mathrm{Eff} = \mathrm{Set}[\mathbf{P}] \end{array}$$

which make Eff a retract of $\mathrm{Set}[\mathbf{R}]$ (a retract of a very special sort since $\bar{i}^* = \bar{r}^*$ so that $\bar{r}^*\bar{i}^* \leqslant \mathrm{id}$).

<u>(iv)</u> Let $\mathbb{A}$ be a combinatory algebra (in Set) and define a new one $m\mathbb{A}$ ("multi-valued $\mathbb{A}$") from it as follows:

$m\mathbb{A}$ has underlying set $D(A) = \{p \subseteq A \mid p \text{ inhabited}\}$ with application given by

$Ep(q)$ iff for all $x \in p$, $y \in q$ $Ex(y)$,

and in this case $p(q) = \{x(y) \mid x \in p \text{ and } y \in q\}$

(thus we may take the K and S combinators for $m\mathbb{A}$ to be $\{K\}$ and $\{S\}$ respectively).

Let $R_{\mathbb{A}}$, $R_{m\mathbb{A}}$ be the realizability triposes on $\mathbb{A}$, $m\mathbb{A}$ with designated truth-values given by the inhabited subsets in each case. Then there is a geometric morphism $i: R_{\mathbb{A}} \longrightarrow R_{m\mathbb{A}}$ given by

$$i^*\Phi = \bigcup \Phi \qquad (\Phi \subseteq D(A)),$$

and $$i_*p = D(p) = \{q \subseteq p \mid q \text{ inhabited}\} \quad (p \subseteq A).$$

(It is not hard to see that i^* and i_* define functors and that i^* is left exact; furthermore $i^* \cdot i_* = id_{PA}$, and if $a \in i^*\Phi \to_{\mathbb{A}} p$, then $\{a\} \in \Phi \to_{m\mathbb{A}} i_*p$.)

<u>4.10 Remarks</u>

(i) Let **C-Trip** denote the (bi)category of C-triposes and geometric morphisms; let **C-Top** denote the (bi)category whose objects are pairs (E, Δ) where E is a topos and $\Delta : C \longrightarrow E$ a left exact functor satisfying condition (ii) of Theorem 3.10, and whose arrows $(E,\Delta) \longrightarrow (E',\Delta')$ are pairs (g,α) where $g: E \longrightarrow E'$ is a geometric morphism and $\alpha : g^* \cdot \Delta' \longrightarrow \Delta$ an isomorphism. Then

$$(P \xrightarrow{f} R) \longmapsto ((C[P], \Delta_P) \xrightarrow{(\bar{f}, \cong)} (C[R], \Delta_R))$$

gives a (bi)functor **C-Trip** $\longrightarrow$ **C-Top**, which by Theorems 3.10 and 4.8 is an equivalence.

(ii) Let **P** and **R** be realizability triposes: let us suppose that they are Set-triposes, given by combinatory algebras $\mathbb{A}$ and $\mathbb{B}$ (designated truth-values being the inhabited subsets of A and B). Let $l : P \longrightarrow R$ be a <u>regular</u> Set-indexed functor, i.e. one which preserves $\top$, $\wedge$ and $\exists$ (for example the inverse image part of a geometric morphism).

Then l is determined up to isomophism by the map $\lambda: A \longrightarrow PB$ sending $a \in A$ to $l\{a\}$. For consider $E_A = \{(a,p) \mid a \in p \subseteq A\}$ and let $\pi, \sigma: E_A \longrightarrow PA$ be $\pi(a,p) = p$, $\sigma(a,p) = \{a\}$; since $\bigvee_P$ is just $\bigcup$ (c.f. 1.5), we have that $\exists\pi(\sigma) = id_{PA}$ and hence

$$l = l(id_{PA}) = l(\exists\pi)\sigma \dashv\vdash \exists\pi(l\sigma) = \exists\pi(\lambda).$$

Thus l is isomorphic to the map $PA \longrightarrow PB$ sending $p \subseteq A$ to $\bigcup\{\lambda(a) \mid a \in p\}$.

Since l preserves $\vdash$ and $\top$ and $\top_1 \vdash_1 \{a\}$, we have

$$\top_1 \vdash_1 l(\top_1) \vdash_1 l\{a\} = \lambda(a),$$

so that:

(a) <u>for each $a \in A$, $\lambda(a) \subseteq B$ is inhabited.</u>

Now let $\Delta = \{(a,a') \mid Ea(a')\}$, $\delta: \Delta \longrightarrow PA$ be $\delta(a,a') = \{a(a')\}$ and $\pi_1, \pi_2: \Delta \longrightarrow PA$ be $\pi_1(a,a') = \{a\}$, $\pi_2(a,a') = \{a'\}$. Then $\pi_1 \wedge \pi_2 \vdash_\Delta \delta$; so since l preserves $\vdash$ and $\wedge$, we have $l\pi_1 \wedge l\pi_2 \vdash_\Delta l\delta$, hence $l\pi_1 \vdash_\Delta l\pi_2 \rightarrow l\delta$ and therefore:

(b) <u>there is $b \in B$ such that for all $a, a' \in A$, if $Ea(a')$ then $b \in \lambda(a) \rightarrow (\lambda(a') \rightarrow \lambda(a(a')))$.</u>

Conversely if $\lambda: A \longrightarrow PB$ satisfies (a) and (b), it is not hard to see that defining $l(p) = \bigcup\{\lambda(a) \mid a \in p\}$ we get a regular Set-indexed functor $l: \mathbf{P} \longrightarrow \mathbf{R}$.

There are very many examples of regular functors between realizability triposes. For example if $\mathbb{A}$ is $\mathbb{N}$ and $\ulcorner n \urcorner \in B$ represents the numeral $n \in \mathbb{N}$ in $\mathbb{B}$ (see 6.7), then $\lambda(n) = \{\ulcorner n \urcorner\}$ satisfies (a) and (b) and so determines a regular functor $\mathbf{P} \longrightarrow \mathbf{R}$. Again, with $\mathbb{A} = \mathbb{N}$ and $\mathbb{B} = \mathbb{G}$ the "graph" model of the λ-calculus ($P\mathbb{N}$ with Scott application) we get a regular functor $l: \mathbf{R} \longrightarrow \mathbf{P}$ by letting

$$\lambda(p) = \{n \mid W_n \subseteq p\},$$

where $W_n = \{m \mid En(m)\}$ is the recursively enumerable subset of $\mathbb{N}$ with index n (and we get another by replacing W_n by e_n, the n^{th} finite subset of $\mathbb{N}$).

5. TOPOLOGIES AND SUB-TRIPOSES

In Chapter 4 we saw how the obvious generalisation of the notion of "continuous map of locales" to triposes gave results about geometric morphisms between the induced toposes. In this chapter we shall do the same for the notion of "nucleus" (or J-operator) on a locale (c.f. [6]) and sheaf subtoposes. We shall confine ourselves to the case when **P** is $E(-,\Sigma)$, a canonically presented tripos on a topos E.

Now a sheaf subtopos of $E[\mathbf{P}]$ corresponds to a Lawvere-Tierney topology $j:\Omega \longrightarrow \Omega$ in $E[\mathbf{P}]$ (c.f. [10], Chapter 3). Given such a map, let the subobject of Ω it classifies (the generic j-dense subobject) be canonically represented by a strict relation $J:\Sigma \longrightarrow \Sigma$ for $\Omega = (\Sigma, \leftrightarrow_{\mathbf{P}})$. Since J is a relation we have

(1) $\mathbf{P} \models (J(\underline{p}) \wedge \underline{p} \to \underline{q}) \to J(\underline{q})$.

Since j is a topology we have

(2) $\mathbf{P} \models J(\top)$,

(3) $\mathbf{P} \models J(J\underline{p}) \leftrightarrow J(\underline{p})$, and

(4) $\mathbf{P} \models J(\underline{p} \wedge \underline{q}) \leftrightarrow (J\underline{p} \wedge J\underline{q})$.

From these it follows that

(5) $\mathbf{P} \models (\underline{p} \to \underline{q}) \to (J\underline{p} \to J\underline{q})$, and

(6) $\mathbf{P} \models \underline{p} \to J\underline{p}$.

In view of (5) J determines (as in Remark 4.4) an E-indexed functor $J:\mathbf{P} \longrightarrow \mathbf{P}$, which by (2) and (4) is left exact, by (6) is <u>inflationary</u> ($id \leqslant J$) and by (3) is <u>idempotent</u> ($JJ \cong J$). Conversely any such functor gives a strict relation on Ω in $E[\mathbf{P}]$ whose classifying map is a Lawvere-Tierney topology.

5.1 Definition

A (Lawvere-Tierney) topology on an E-tripos $\mathbf{P}$ is an inflationary, idempotent left exact E-indexed functor $J:\mathbf{P} \to \mathbf{P}$.

From the above, such topologies correspond to sheaf subtoposes of $E[\mathbf{P}]$.

5.2 Remarks

(i) Such a topology on $\mathbf{P} = E(-,\Sigma)$ can be specified by a map $J: \Sigma \longrightarrow \Sigma$ satisfying just

$$\mathbf{P} \models (\underline{p} \to \underline{q}) \to (J\underline{p} \to J\underline{q}),$$

$$\mathbf{P} \models J(\top),$$

and $$\mathbf{P} \models J(J\underline{p}) \to J(\underline{p}),$$

since (4) and (6) follow from these.

(ii) Let $(E[\mathbf{P}])_j$ be the sheaf subtopos corresponding to a topology J on $\mathbf{P}$, with associated sheaf functor $L:E[\mathbf{P}] \longrightarrow (E[\mathbf{P}])_j$. Applying Theorem 3.10 to $L\circ\Delta_{\mathbf{P}}:E \longrightarrow (E[\mathbf{P}])_j$, we see that $(E[\mathbf{P}])_j$ is eqivalent (over E) to $E[\mathbf{P}_J]$, for some canonically presented E-tripos, $\mathbf{P}_J$. We may describe $\mathbf{P}_J$ as follows:

The underlying E-indexed category of $\mathbf{P}_J$ is just $E(-,\Sigma)$, but we redefine $\to$, $\forall$ and $\mathbf{D}$ by letting

$$\to_{\mathbf{P}_J} \text{ be } \Sigma\times\Sigma \xrightarrow{id\times J} \Sigma\times\Sigma \xrightarrow{\to_{\mathbf{P}}} \Sigma\,,$$

$(\forall f)\varphi$ be $(\forall f)(J\varphi)$ (so that $\bigwedge_{\mathbf{P}_J}$ is $P\Sigma \xrightarrow{PJ} P\Sigma \xrightarrow{\bigwedge_{\mathbf{P}}} \Sigma$ if $\mathbf{P}$ has fibre-wise quantification), and

$$1 \xrightarrow{p} \Sigma \text{ be in } \mathbf{D}_{\mathbf{P}_J} \text{ iff } 1 \xrightarrow{p} \Sigma \xrightarrow{J} \Sigma \text{ is in } \mathbf{D}_{\mathbf{P}}.$$

Thus $\overset{\mathbf{P}_J}{\vdash}$ is given by $\varphi \overset{\mathbf{P}_J}{\vdash} \psi$ iff $\varphi \overset{\mathbf{P}}{\vdash} J\psi$, whilst $\top, \wedge, \bot, \vee$ and $\exists$ remain unchanged.

(iii) Let $f:\mathbf{P} \longrightarrow \mathbf{R}$ be a geometric morphism between E-triposes, inducing $\bar{f}:E[\mathbf{P}] \longrightarrow E[\mathbf{R}]$ as in Proposition 4.7. Then $J = f_* f^*$ is a topology on $\mathbf{R}$ and the surjection-inclusion factorisation of $\bar{f}$ (c.f. 4.14 of [10]) takes the form

$$\bar{f}:\mathrm{E}[\mathbf{P}] \longrightarrow \mathrm{E}[\mathbf{R}_J] \hookrightarrow \mathrm{E}[\mathbf{R}].$$

Thus $\bar{f}$ is an inclusion iff $f^* f_* \cong \mathrm{id}$, and in this case we shall say that f is an <u>inclusion</u>.

<u>5.3 Examples</u>

<u>(i)</u> Topologies on the canonical triposes of locales are just nuclei (or J-operators) on the locales themselves.

<u>(ii)</u> Consider the geometric morphism $\eta:\mathbf{Sub}_E \longrightarrow \mathbf{P}$ of Example 4.9(ii). We noted there that $\eta^*\eta_* = \gamma\circ\delta = \mathrm{id}_\Omega$, so η is an inclusion: the corresponding topology, which we shall denote by $J_0:PA \longrightarrow PA$ sends $\underline{p}\subseteq A$ to $J_0(\underline{p}) = \{\underline{a} \mid \underline{p}$ is inhabited$\}$.

Note that when E is Set, J_0 is $\neg\neg$; so in particular $(\mathrm{Eff})_{\neg\neg} \simeq \mathrm{Set}$. Also note that unless $\mathbb{A}$ is the degenerate combinatory algebra $\mathbb{0}$, E is always a proper subtopos of $\mathrm{E}[\mathbf{P}]$ (i.e. J_0 is not the least topology id_{PA}).

(iii) Corresponding to the inclusion $i:\mathbf{P} \hookrightarrow \mathbf{R}$ of Example 4.9(iii) is the topology $J(R) = \{(m,n) \mid (n,n)\in R$ and $(m,m)\in R\}$. Thus $\mathrm{Eff} \simeq \mathrm{Set}[\mathbf{R}_J]$.

(iv) We noted in 4.9(iv) that $i^* i_* = \mathrm{id}_{PA}$, so that i is an inclusion $\mathbf{R}_{\mathbb{A}} \hookrightarrow \mathbf{R}_{m\mathbb{A}}$. As in (ii), unless $\mathbb{A}$ is $\mathbb{0}$, i is a strict inclusion. For the topology corresponding to i sends $\Phi \subseteq D(A)$ to $J(\Phi) = \{q = \bigcup\Phi \mid q$ is inhabited$\}$; thus if $K \neq S$ in A, then $J\Phi_1 = J\Phi_2$ and $\Phi_1 \cap \Phi_2 = \emptyset$ where $\Phi_1 = \{\{K,S\}\}$ and $\Phi_2 = \{\{K\},\{S\}\}$, so that $\bigcap\{J\Phi \to \Phi \mid \Phi \subseteq D(A)\}$ is empty, i.e. J is not isomorphic to the least topology, id.

We are now going to explore the subtoposes of the effective topos. To do this we require some generalities on generating topologies in a tripos:

<u>5.4 Lemma</u> (c.f. 3.57 of [10])

Let $\mathbf{P} = \mathrm{E}(-,\Sigma)$ be a canonically presented E-tripos. If

$m:\Sigma \longrightarrow \Sigma$, let $J_m:\Sigma \longrightarrow \Sigma$ be given by

$$J_m(\underline{p}) = [\![\forall \underline{q}((m\underline{q} \to \underline{q}) \wedge (\underline{p} \to \underline{q}) \to \underline{q})]\!].$$

Then

(i) if m satisfies $\mathbf{P} \models (\underline{p} \to \underline{q}) \to (m\underline{p} \to m\underline{q})$, then J_m is a topology on $\mathbf{P}$; moreover $m \vdash_\Sigma J_m$ and J_m is the least such topology;

(ii) if $(X,=)$ is a $\mathbf{P}$-object and $R:X \longrightarrow \Sigma$ a strict relation on $(X,=)$, then the least topology J which makes $\|R\| \rightarrowtail (X,=)$ J-dense is J_m, where $m:\Sigma \longrightarrow \Sigma$ is

$$m(\underline{p}) = [\![\exists \underline{x} \in X(R(\underline{x}) \to \underline{p})]\!].$$

Proof

(i) Straightfoward deductions and use of the Soundness Lemma of 2.1.

(ii) $\|R\| \rightarrowtail (X,=)$ is J-dense iff $\mathbf{P} \models \forall \underline{x} \in X\ J(R\underline{x})$. But a simple logical deduction shows that if $m':\Sigma \longrightarrow \Sigma$ satisfies $\mathbf{P} \models (\underline{p} \to \underline{q}) \to (m'\underline{p} \to m'\underline{q})$, then

$$m \vdash_\Sigma m' \quad \text{iff} \quad \mathbf{P} \models \forall \underline{x} \in X\ m'(R\underline{x}).$$

Then since by (i) J_m is the least topology J satisfying $m \vdash_\Sigma J$, the result follows.

□

Notation: write $\leqslant$ for the (pre-)order on topologies (given by $\vdash_\Sigma$) and $<$ for the strict order. We will denote the least and greatest topologies by $J_\perp$ and $J_\top$ respectively; thus $J_\perp(p) = p$ and $J_\top(p) = \top$.

Now let E be Set and $\mathbf{P}$ the effective tripos. So far we know of three distinct topologies on $\mathbf{P}$, namely $J_\perp, J_\top$ and the topology $J_0 = \neg\neg$ of 5.3(ii), and we have $J_\perp < \neg\neg < J_\top$.

5.5 Lemma

Let $m:P\mathbb{N} \longrightarrow P\mathbb{N}$ satisfy $\mathbf{P} \models (\underline{p} \to \underline{q}) \to (m\underline{p} \to m\underline{q})$, and J_m

be as in 5.4. Then the following are equivalent:

(i) $J_m < J_\top$,

(ii) $J_m \leqslant \neg\neg$,

(iii) $m(\bot) = \bot$.

Furthermore we have

(iv) $J_\bot < J_m$ iff $\bigcap \{mp \to p \mid p \subseteq \mathbb{N}\}$ is empty,

and for any topology J

(v) $\neg\neg \leqslant J$ iff $\bigcap \{J\{n\} \mid n \in \mathbb{N}\}$ is inhabited.

Proof

$J_\top \leqslant J_m$ iff $P \models \forall \underline{p}\ J_m(\underline{p})$ iff $J_m(\bot)$ inhabited iff $m(\bot)$ is inhabited. So we have (i) iff (iii). But (ii) implies that $m \leqslant J_m \leqslant \neg\neg$ so $m(\bot) \vdash_1 \neg\neg(\bot) = \bot$, i.e. $m(\bot) = \bot$; conversely if $m(\bot) = \bot$, then $J_m(\bot) = \bot$, so $\lambda x.x \in J_m(p) \to \neg\neg(p)$, all $p \subseteq \mathbb{N}$, i.e. $J_m \leqslant \neg\neg$. Thus (ii) iff (iii).

Since J_m is not isomorphic to $J_\bot$ iff $\bigcap \{mp \to p \mid p \subseteq \mathbb{N}\}$ is empty, (iv) is immediate.

For (v), note that $\neg\neg \leqslant J$ iff $\bigcap \{J(p) \mid p \text{ is inhabited}\}$ is inhabited, which certainly implies $\bigcap \{J\{n\} \mid n \in \mathbb{N}\}$ is inhabited. Conversely, suppose $a \in \bigcap_n J\{n\}$ and $p \subseteq \mathbb{N}$ is inhabited, say $n \in p$; then $e = \lambda x.x \in \{n\} \to p$, so $b(e) \in J\{n\} \to Jp$, where b is any element of $\bigcap_{p,q} (p \to q) \to (Jp \to Jq)$. Thus $b(e)(a) \in Jp$, all inhabited $p \subseteq \mathbb{N}$.

□

Remark

M. Hyland has shown that, quite suprisingly $\bigcap_n J\{n\}$ is inhabited iff $J\{0\} \cap J\{1\}$ is. Since (by 5.4(ii)) the latter is true iff $1 + 1 \rightarrowtail \Delta_P(1 + 1)$ is J-dense, we see that $\neg\neg$ is the least topology that makes $1 + 1 \rightarrowtail \Delta_P(1 + 1)$ dense. In other words, if we force $\Delta : \text{Set} \longrightarrow \text{Eff}$ to preserve finite coproducts, we collapse Eff back down to Set.

Lemma 5.5 shows that there are no topologies on Eff strictly between $\neg\neg$ and $J_\top$; are there any between $J_\bot$ and $\neg\neg$? It turns out that there are very many, but a little work is required to exhibit them.

5.6 Proposition

Let (X,=), R, m and J_m be as in Lemma 5.4(ii). Suppose further that for each x in X, R(x) is inhabited. Let $* \in \mathbb{N}$ be any index for the empty partial function, and define $m^*: P\mathbb{N} \longrightarrow P\mathbb{N}$ by

$$m^*(p) = \bigcap \{ q \subseteq \mathbb{N} \mid p \wedge \{*\} \subseteq q \supseteq m(q)\}.$$

Then $m^* \dashv\vdash J_m$.

Proof

If $p \subseteq q$, then $m(p) \subseteq m(q)$; hence

$$\begin{aligned} m(m^*p) &\subseteq \bigcap \{m(q) \mid p \wedge \{*\} \subseteq q \supseteq m(q)\} \\ &\subseteq \bigcap \{ q \mid p \wedge \{*\} \subseteq q \supseteq m(q)\} \\ &= m^*(p). \end{aligned}$$

Thus $P \models \forall \underline{p}(m(m^*\underline{p}) \to m^*\underline{p})$, and certainly $P \models \forall \underline{p}(\underline{p} \to m^*\underline{p})$. Then since $J_m(\underline{p}) \models^P_{\underline{p}} (m(m^*\underline{p}) \to m^*\underline{p}) \wedge (\underline{p} \to m^*\underline{p}) \to m^*\underline{p}$, we get $J_m \vdash m^*$.

To show conversely that $m^* \vdash J_m$ we must find a number in

$$\bigcap_{p,q} (mq \to q) \wedge (p \to q) \to (m^*q \to p).$$

Now by the Recursion Theorem (see 6.6), there is an f in $\mathbb{N}$ such that for all $a,b,n \in \mathbb{N}$, $f\langle a,b\rangle$ is defined and satisfies

$$f\langle a,b\rangle(n) \equiv \begin{cases} b(n)_0 & \text{if } (n)_1 = *, \\ a\langle (n)_0, f\langle a,b\rangle \circ (n)_1 \rangle & \text{otherwise.} \end{cases}$$

(where, as in the Remark in 1.5, $\langle\cdot,\cdot\rangle$, $(\cdot)_0$ and $(\cdot)_1$ denote pairing and unpairing, and $f\langle a,b\rangle \circ (n)_1$ denotes the composition of $(n)_1$ with $f\langle a,b\rangle$, i.e. $\lambda m.f\langle a,b\rangle((n)_1 m)$).

Given subsets p,q of $\mathbb{N}$ and numbers $a \in m(q) \to q$, $b \in p \to q$, consider the set $r = \{n \in m^*(p) \mid f\langle a,b\rangle(n) \in q\}$. By definition $f\langle a,b\rangle \in r \to q$. If $n \in m(r)$, say $(n)_0 \in [\![x \in X]\!]$ and $(n)_1 \in R(x) \to r$

some $x \in X$, then $f<a,b>\circ(n)_1 \in R(x) \to q$ so that $m(q)$ contains $<(n)_0, f<a,b>\circ(n)_1>$ and thus $a<(n)_0, f<a,b>\circ(n)_1>$ is in q. But since $R(x)$ is inhabited and $(n)_1 \in R(x) \to r$, $(n)_1 \neq *$, so that by definition of f, $f<a,b>(n) = a<(n)_0, f<a,b>\circ(n)_1>$ is in q. Since $n \in m(r) \subseteq m(m^*p) \subseteq m^*p$, we conclude that $n \in r$.

Thus $m(r) \subseteq r$. But also $p \wedge \{*\} \subseteq r$ (since if $n \in p \wedge \{*\}$, then certainly $n \in m^*p$ and since $(n)_1 = *$, $f<a,b> = b(n)_0 \in q$). So by definition of m^*, $m^*(p) \subseteq r$ and therefore $m^*(p) = r$. Thus $f<a,b> \in m^*(p) \to q$, and therefore

$$f \in (mq \to q) \wedge (p \to q) \to (m^*p \to q)$$

all $p,q \subseteq N$.

□

5.7 Corollary

Every topology J on the effective tripos is isomorphic to the m^* for some $(X,=)$ and strict relation R on $(X,=)$.

Proof

Since J is the least topology which inverts the generic J-dense subobject, J is isomorphic to the J_m for some $(X,=)$ and R.

If $J \cong J_\top$, then $J \dashv\vdash m^*$ for $(X,=)$ equal to $\Delta(1)$ and $R: 1 \longrightarrow P\mathbb{N}$ corresponding to $\emptyset$ in $P\mathbb{N}$. So we may suppose that $J \cong J_m < J_\top$, i.e. (by 5.5) that $m(\bot) = \bot$, which means that if $R(x) = \bot$ then $[\![x \in X]\!] = \bot$. Then $\|R\| \rightarrowtail (X,=)$ is isomorphic to $\|R|_{X_0}\| \rightarrowtail (X_0, =|_{X_0})$ where X_0 is $\{x \mid [\![x \in X]\!]$ is inhabited$\}$. Now we can apply Proposition 5.6 and get that $J \cong J_m \dashv\vdash m^*$.

□

5.8 Example

Using Proposition 5.6, we can give a topology J on the effective tripos such that $J_\bot < J < \neg\neg$.

Take $(X,=)$ to be $\Delta(\mathbb{N})$, $R:\mathbb{N} \longrightarrow P\mathbb{N}$ to be $R(n) = \{x \mid n \leqslant x\}$, and J to be J_m for m given as in 5.4(ii). Thus

$$m(p) = [\![\exists x \in X(Rx \to p)]\!] = \{a \mid \exists n \forall x \geqslant n\ ((a)_1 x \in p)\}.$$

Now by 5.5 $J_\perp < J_m \leqslant \neg\neg$ (for $m(\perp) = \perp$, and $\bigcap_p mp \to p = \perp$ since $< 0, \lambda x.x > \in \bigcap_n m\{x \mid n \leqslant x\}$ whilst $\bigcap_n \{x \mid n \leqslant x\} = \perp$).

To show that $J_m < \neg\neg$, since by 5.6 $J_m \dashv\vdash m^*$, we must show (5.5(v)) that $\bigcap_n m^*\{n\} = \perp$. But in fact $m^*\{0\} \cap m^*\{1\} = \perp$. This follows since

(a) m preserves $\cap$,

(b) $< n,* > \notin m(p)$, any $p \subseteq \mathbb{N}$, and

(c) $m^*p = m(m^*p) \cup p \wedge \{*\}$.

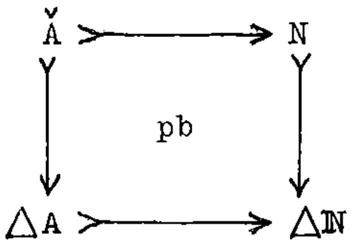
From these we get $< 0,* > \in m^*\{0\} \backslash m^*\{1\} \supseteq m(m^*\{0\} \backslash m^*\{1\})$, so that by definition of $m^*\{0\}$, it is contained in $m^*\{0\} \backslash m^*\{1\}$, i.e. $m^*\{0\}$ and $m^*\{1\}$ are disjoint, as required. Therefore J_m is a topology for the effective tripos lying strictly between $J_\perp$ and $\neg\neg$.

□

In fact using Proposition 5.6, one can show that the lattice of topologies between $J_\perp$ and $\neg\neg$ is extremely rich. As an example we quote the following:

<u>Theorem</u> (M. Hyland; W. Powell)

Let $N \rightarrowtail \Delta\mathbb{N}$ be the canonical monic in Eff given by the predicate $n \longmapsto \{n\}$ (we shall see in Chapter 6 that N is actually a natural number object in the topos Eff). Then each subset A of $\mathbb{N}$ determines a subobject $\check{A} \rightarrowtail N$ by restriction:



Let J_A be the least topology that forces $\check{A} \rightarrowtail N$ to be decidable. Then the map $A \longmapsto J_A$ induces an embedding of the $\vee$-semilattice of Turing Degrees into the Heyting algebra of topologies on the effective topos.

□

6. ITERATION

Let us consider iterating the constuction of Chapter 2. Suppose then that C is a finitely complete category, **P** a C-tripos and **R** a C[**P**]-tripos. So we have left exact functors

$$C \xrightarrow{\Delta_{\mathbf{P}}} C[\mathbf{P}] \xrightarrow{\Delta_{\mathbf{R}}} C[\mathbf{P}][\mathbf{R}],$$

and ask whether their composite arises from a C-tripos. By Theorem 3.10 the answer is yes, if we can verify the condition (ii) of that theorem for $\Delta_{\mathbf{R}}\circ\Delta_{\mathbf{P}}$.

<u>6.1 Lemma</u>

Let C be finitely complete, E,F be toposes, $\Delta:C \longrightarrow E$ and $\Delta':E \longrightarrow F$ left exact functors, and suppose also that Δ' preserves epimorphisms. Then if Δ and Δ' satisfy condition (ii) of Theorem 3.10, so does $\Delta'\circ\Delta$.

<u>Proof</u>

Given an object B of F, we have epimorphisms $\beta'_B:\Delta'(\hat{B}) \twoheadrightarrow \tilde{B}$ in F and $\beta_{\hat{B}}:\Delta(\hat{\hat{B}}) \twoheadrightarrow \tilde{\hat{B}}$ in E, classifying partial maps

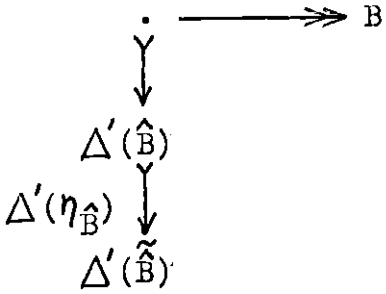
$$\begin{array}{ccc} \cdot & \twoheadrightarrow & B \\ \downarrow & & \\ \Delta'(\hat{B}) & & \end{array} \quad \text{in F and} \quad \begin{array}{ccc} \cdot & \twoheadrightarrow & \hat{B} \\ \downarrow & & \\ \Delta(\hat{\hat{B}}) & & \end{array} \quad \text{in E}$$

(which present B and B' as subobjects of constant objects in a weakly universal way). Then the map $\beta''_B:\Delta'\Delta(\hat{\hat{B}}) \longrightarrow \tilde{B}$ classifying

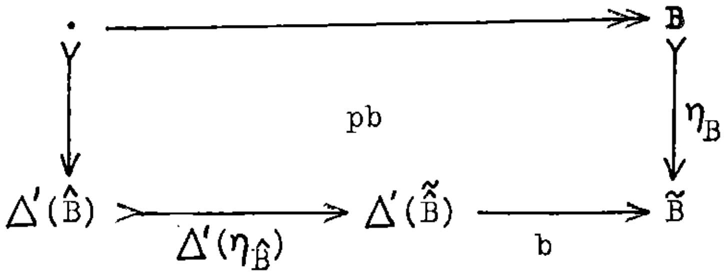
$$\begin{array}{ccccc} \cdot & \longrightarrow & \cdot & \twoheadrightarrow & B \\ \downarrow & \text{pb} & \downarrow & & \\ \Delta'(\cdot) & \longrightarrow & \Delta'(\hat{B}) & & \\ \downarrow & & & & \\ \Delta'(\Delta(\hat{\hat{B}})) & & & & \end{array}$$

satisfies (ii)(a) for B.

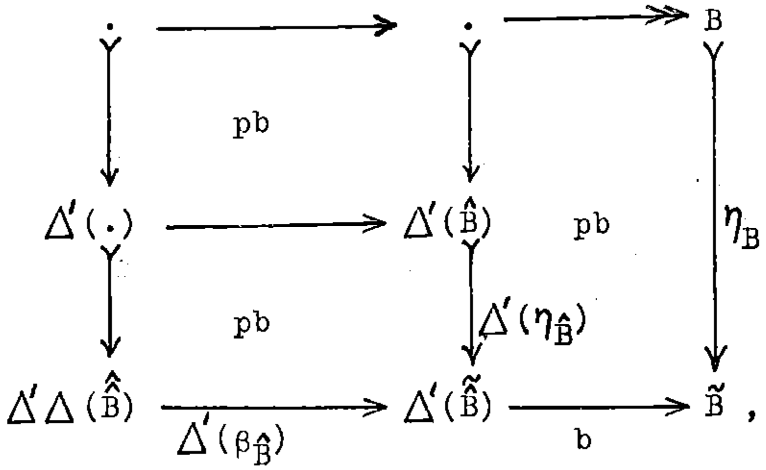
Now if $b: \Delta'(\tilde{\hat{B}}) \longrightarrow \tilde{B}$ classifies



then we have pullback squares



and



so that $\beta'_B = b \circ \Delta'(\eta_{\hat{B}})$ and $\beta''_B = b \circ \Delta'(\beta_{\hat{B}})$. Then since β'_B is epi, so is b; and since β_B is epi and Δ' preserves epimorphisms, $\Delta'(\beta_{\hat{B}})$ is also epi. Therefore β''_B is an epimorphism, and hence (ii)(b) is satisfied as well.

$\square$

<u>6.2 Theorem</u>

Let C be a finitely complete category, **P** a C-tripos and **R** a C[**P**]-tripos with fibre-wise quantification. Then $\mathbf{R} \circ \Delta_{\mathbf{P}}^{op}$ is a C-tripos and $\Delta_{\mathbf{R} \circ \Delta_{\mathbf{P}}^{op}}: C \longrightarrow C[\mathbf{R} \circ \Delta_{\mathbf{P}}^{op}]$ is equivalent (over C) to $C \xrightarrow{\Delta_{\mathbf{P}}} C[\mathbf{P}] \xrightarrow{\Delta_{\mathbf{R}}} C[\mathbf{P}][\mathbf{R}]$.

Proof

By Proposition 3.13, $\Delta_{\mathbf{R}}$ preserves epimorphisms; so by Lemma 6.1 we can apply Theorem 3.10 to $\Delta_{\mathbf{R}}\circ\Delta_{\mathbf{P}}$ and get that $C[\mathbf{P}][\mathbf{R}]$ is equivalent over C to $C[\mathbf{S}]$ for some C-tripos $\mathbf{S}$. But by Proposition 3.6, $\mathbf{S}$ is equivalent to $\mathbf{Sub}_{C[\mathbf{P}][\mathbf{R}]}(\Delta_{\mathbf{R}}\circ\Delta_{\mathbf{P}})^{op}$ which is itself equivalent to $\mathbf{R}\circ\Delta_{\mathbf{P}}^{op}$.

$\square$

6.3 Remark

If E is a topos and $\mathbf{P} = E(-,\Sigma)$ and $\mathbf{R} = E[\mathbf{P}](-,(\Lambda,=))$ are canonically presented triposes, let us see how to canonically present the tripos $\mathbf{R}\circ\Delta_{\mathbf{P}}^{op}$.

Recall from Lemma 3.7 that we can take $\beta_{(\Lambda,=)}:\Delta_{\mathbf{P}}(\hat{\Lambda}) \twoheadrightarrow \widetilde{(\Lambda,=)}$ in $E[\mathbf{P}]$ to be the classifying map of

$$\begin{array}{ccc} \| S_\Lambda \| & \xrightarrow{\;b_\Lambda\;} & (\Lambda,=) \\ {\scriptstyle i_\Lambda}\big\downarrow & & \\ \Delta_{\mathbf{P}}(\Sigma^\Lambda) & & \end{array}$$

where $S_\Lambda:\Sigma^\Lambda \longrightarrow \Sigma$ is $S_\Lambda(\underline{R}) = [\![\exists \underline{x} \in \Lambda\, \forall \underline{x}'(R\underline{x}' \leftrightarrow \underline{x}' = \underline{x})]\!]$. Now b_Λ is a predicate in $\mathbf{R}(\| S_\Lambda \|)$: existentially quantifying along i_Λ (for $\mathbf{R}$) gives us a predicate $\sigma = \exists^{\mathbf{R}} i_\Lambda(b_\Lambda)$ in $\mathbf{R}(\Delta_{\mathbf{P}}(\Sigma^\Lambda))$. We shall show that σ is a generic predicate for $\mathbf{R}\circ\Delta_{\mathbf{P}}^{op}$ (c.f. (c') of 1.2).

Given $\varphi \in \mathbf{R}(\Delta_{\mathbf{P}} I)$, by Lemma 3.7 there is a map $f:I \longrightarrow \Sigma$ in E such that $\varphi = \beta_{(\Lambda,=)}\circ\Delta_{\mathbf{P}} f$, i.e. we have

$$\begin{array}{ccccc} \Delta_{\mathbf{P}} I & & \xrightarrow{\quad\varphi\quad} & & (\Lambda,=) \\ & \overset{\varphi'}{\dashrightarrow} & \| S_\Lambda \| & \xrightarrow{\;b\;} & \\ {\scriptstyle id}\big\downarrow & \text{pb} & \big\downarrow{\scriptstyle i_\Lambda} & & \\ \Delta_{\mathbf{P}} I & \xrightarrow{\Delta_{\mathbf{P}}(f)} & \Delta_{\mathbf{P}}(\Sigma^\Lambda). & & \end{array}$$

Then (using the Beck condition on the pullback square) we have

$$\sigma \circ \Delta_P(f) = R(\Delta_P f)\exists(i_\Lambda)b_\Lambda \dashv\vdash \exists(id)R(\varphi')b_\Lambda \dashv\vdash b_\Lambda \cdot \varphi' = \varphi$$

as required. Thus the structure on $R \cdot \Delta_P^{op}$ induces a tripos structure on $\mathbb{E}(-,\Sigma^\Lambda)$ and $R \cdot \Delta_P^{op} \simeq \mathbb{E}(-,\Sigma^\Lambda)$.

If the only triposes we knew were localic ones, the content of Theorem 6.2 would be trivial: merely that the composite of two localic geometric morphisms is again localic. However by iterating realizability triposes or mixtures of localic and realizability triposes, Theorem 6.3 enables us to construct new examples. The rest of this chapter will be concerned with illustrating this.

<u>6.4 Example</u>

We have already met an example of a "realizability followed by locale" iteration, namely the tripos $\mathbf{R}$ of 1.6. For in 4.9(iii) we showed that Eff is a retract of Set[$\mathbf{R}$]:

$$\begin{array}{ccc} \mathrm{Eff} & \xrightarrow{\ \bar{I}\ } & \mathrm{Set}[\mathbf{R}] \\ & \searrow^{id} & \downarrow \bar{r} \\ & & \mathrm{Eff}. \end{array}$$

Now every $\mathbf{R}$-object is the subquotient of some constant object $\Delta_{\mathbf{R}}(I)$, and $\Delta_{\mathbf{R}}(I) \cong \bar{r}^*(\Delta I)$ (ΔI the constant object on I in Eff). So $\bar{r}$ is a localic geometric morphism and hence Set[$\mathbf{R}$] is equivalent to the localic extension of Eff by the Eff-locale $\bar{r}_*(\Omega_{\mathrm{Set}[\mathbf{R}]})$. Recalling that $\mathbf{R} = \mathrm{Set}(-,\Sigma)$ where Σ is $\{R \subseteq \mathbb{N}\times\mathbb{N} \mid R \text{ is symmetric and transitive}\}$, since $r_* = u$ preserves $\exists$, $\bar{r}^*$ may be calculated directly from u: thus $\bar{r}_*(\Omega_{\mathrm{Set}[\mathbf{R}]})$ is Σ with the equality

$$\Sigma\times\Sigma \xrightarrow{\ \leftrightarrow_{\mathbf{R}}\ } \Sigma \xrightarrow{\ u\ } P\mathbb{N}\,.$$

6.5 Example: realizability and forcing

Our next example (due to M. Hyland, from a suggestion of P. Aczel) is of a "locale followed by realizability" iteration. The locale will be $\Omega(\mathbb{N}^{\mathbb{N}})$, the open subsets of Baire space ($=\mathbb{N}^{\mathbb{N}}$ with the product topology). (In fact we will really be dealing with the locale of "formal Baire space" as in [5], but since we shall give the example over Set the distinction is masked by the Axiom of Choice.)

Let $\mathbb{N}^{<\mathbb{N}}$ be the set of finite sequences of numbers, partially ordered by the relation of <u>extension</u>. Write $v \supset u$ if the sequence v extends the sequence u; also write $\alpha \supset u$ if the infinite sequence $\alpha \in \mathbb{N}^{\mathbb{N}}$ extends u. Thus the sets

$$B(u) = \{\alpha \in \mathbb{N}^{\mathbb{N}} \mid \alpha \supset u\} \qquad (u \in \mathbb{N}^{<\mathbb{N}})$$

form a basis for the topology on $\mathbb{N}^{\mathbb{N}}$.

If S is a subset of $\mathbb{N}^{<\mathbb{N}}$ and u is a finite sequence, if

(a) $v \in S$ implies $v \supset u$, and

(b) $B(u) = \bigcup\{B(v) \mid v \in S\}$,

then we shall say that <u>S covers u</u>; let C(u) be the set of covers of u.

The realizability part of our example will use recursion relative to a partial function $\mathbb{N} \longrightarrow \mathbb{N}$ on the natural numbers. We will denote by $\{n\}^f(m)$ the value at m (if defined) of the partial recursive functional with index n asking values of $f:\mathbb{N} \longrightarrow \mathbb{N}$. Note that we can regard each u in $\mathbb{N}^{<\mathbb{N}}$ as a partial function and so speak of $\{n\}^u(m)$.

Now in $\mathrm{Set}[\Omega(\mathbb{N}^{\mathbb{N}})] = \mathrm{Sh}(\mathbb{N}^{\mathbb{N}})$ there is a generic sequence of natural numbers given by the function $g:\Delta\mathbb{N} \longrightarrow \Delta\mathbb{N}$ whose graph $\|G\| \rightarrowtail \Delta(\mathbb{N}\times\mathbb{N})$ is canonically represented by the predicate $G:\mathbb{N}\times\mathbb{N} \longrightarrow \Omega(\mathbb{N}^{\mathbb{N}})$ sending (m,n) to $\{\alpha \mid \alpha(n)=m\}$. Let $\mathfrak{G}$ be the combinatory algebra in $\mathrm{Sh}(\mathbb{N}^{\mathbb{N}})$ with underlying sheaf the natural number object $\Delta\mathbb{N}$ and application

$$\underline{n},\underline{m} \longmapsto \{\underline{n}\}^{g}(\underline{m}).$$

If we calculate the predicate $\mathbb{N}\times\mathbb{N}\times\mathbb{N} \longrightarrow \Omega(\mathbb{N}^{\mathbb{N}})$ which canonically represents the graph of this partial application, we find that it is

(1) $\quad \mathrm{App}^{g}(n,m,k) = \bigcup \{B(u) \mid u \in \mathbb{N}^{<\mathbb{N}} \text{ and } \{n\}^{u}(m) = k\}.$

(This is because internally $\{\underline{n}\}^{g}(\underline{m}) = \underline{k} \leftrightarrow \exists \underline{u}(g \supset \underline{u} \wedge \{\underline{n}\}^{\underline{u}}(\underline{m}) = \underline{k}\}$ holds, whilst $\underline{n},\underline{m},\underline{u} \longmapsto \{\underline{n}\}^{\underline{u}}(\underline{m})$ is recursive in $\underline{n},\underline{m}$ and (suitably coded) $\underline{u}$, and like $\mathbf{N}$, $\mathbf{N}^{<\mathbf{N}}$ is the constant sheaf $\Delta(\mathbb{N}^{<\mathbb{N}})$ with $[\![g \supset u]\!] = B(u)$.)

Let $\mathbf{R}$ be a realizability tripos based on $\mathbb{G}$, and consider

$$\mathrm{Set} \xrightarrow{\ \Delta\ } \mathrm{Sh}(\mathbb{N}^{\mathbb{N}}) \xrightarrow{\ \Delta_{\mathbf{R}}\ } \mathrm{Sh}(\mathbb{N}^{\mathbb{N}})[\mathbf{R}],$$

to which we can apply Theorem 6.2. Now in view of the natural isomorphisms

$$\begin{aligned} \mathbf{R}(\Delta I) = \mathrm{Sh}(\mathbb{N}^{\mathbb{N}})(\Delta I, \Omega^{\Delta\mathbb{N}}) &\cong \mathrm{Sh}(\mathbb{N}^{\mathbb{N}})(\Delta(I\times\mathbb{N}), \Omega) \\ &\cong \mathrm{Set}(I\times\mathbb{N}, \Omega(\mathbb{N}^{\mathbb{N}})) \\ &\cong \mathrm{Set}(I, \Omega(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}), \end{aligned}$$

we see that $\mathrm{Sh}(\mathbb{N}^{\mathbb{N}})[\mathbf{R}]$ arises from the tripos structure induced on $\mathrm{Set}(-,\Omega(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}})$ by them. However we will replace $\Omega(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$ by another set Σ for which the tripos structure has an interesting description.

In fact let Σ comprise those subsets p of $\mathbb{N}^{<\mathbb{N}}\times\mathbb{N}$ satisfying

(2) for all $n\in\mathbb{N}$, $u,v\in\mathbb{N}^{<\mathbb{N}}$, if $v\supset u$ and $u \Vdash n\in p$ then $v \Vdash n\in p$,

where instead of "$(u,n)\in p$" we write "$u \Vdash n \in p$" (and read "u forces n to be in p"). Define maps $i:\Omega(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}} \longrightarrow \Sigma$ and $r:\Sigma \longrightarrow \Omega(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$ by putting

(3) $\quad i(\varphi) = \{(u,n) \mid B(u) \subseteq \varphi(n)\}$ and $rp(n) = \{\alpha \mid \exists u(\alpha \supset u \wedge u \Vdash n \in p)\}$

Thus $r\circ i = \mathrm{id}$, and hence the Set-indexed category structure on $\mathrm{Set}(-,\Sigma)$ induced via $r_{*}:\mathrm{Set}(-,\Sigma) \longrightarrow \mathrm{Set}(-,\Omega(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}})$ makes the latter into an equivalence. Write $\mathbf{P}$ for the tripos $\mathrm{Set}(-,\Sigma)$ (so that $\mathrm{Sh}(\mathbb{N}^{\mathbb{N}})[\mathbf{R}] \simeq \mathrm{Set}[\mathbf{P}]$.

If $p,q \in \Sigma$, recalling the definition of $\rightarrow$ for realizability triposes, we have that $(rp \rightarrow_{\mathbf{R}} rq) : \mathbb{N} \longrightarrow \Omega(\mathbb{N}^{\mathbb{N}})$ sends $a \in \mathbb{N}$ to

$$[\![\forall n \;\; rp(n) \rightarrow \exists m\,(\mathrm{App}^{g}(a,n,m) \wedge rq(m))]\!].$$

Now by (1) and (3) we have

$$[\![\exists m\,(\mathrm{App}^{g}(a,n,m) \wedge rq(m)]\!] = \bigcup_{m} \left(\bigcup_{v \Vdash m \in q} B(v) \right) \cap \left(\bigcup_{\{a\}^{w}(n) = m} B(w) \right)$$

$$= \bigcup \{ B(v) \mid v \Vdash \{a\}^{v}(n) \in q \},$$

since m is unique if it exists and "$\{a\}^{w}(n) = m$" enjoys the property (2) that "$v \Vdash m \in q$" does. Thus (u,a) is in $p \rightarrow_{\mathbf{P}} q = i(rp \rightarrow_{\mathbf{R}} rq)$ iff for all $n \in \mathbb{N}$

$$B(u) \wedge rp(n) = \bigcup \{ B(v) \mid \; v \Vdash \; \{a\}^{v}(n) \in q \}.$$

We can rewrite this using the notion of cover defined above:

(4) $u \Vdash a \in (p \rightarrow_{\mathbf{P}} q)$ iff $\forall n, v \supset u (v \Vdash n \in p \Rightarrow \exists\, S \in C(v)\, \forall w \in S (w \Vdash \{a\}^{w}(n) \in q))$

Similarly, using the description of $\bigwedge$ for realizability triposes in 1.5, we find that $\bigwedge_{\mathbf{P}} : P\Sigma \longrightarrow \Sigma$ can be defined by

(5) $u \Vdash a \in \bigwedge_{\mathbf{P}} \Phi$ iff $\forall n\, \forall p \in \Phi\; \exists\, S \in C(u)\, \forall v \in S (v \Vdash \{a\}^{v}(n) \in p\,)$.

To complete the description of the tripos structure on $\mathbf{P}$ we need to know exactly how the designated truth-values for $\mathbf{R}$ were given. If they were given as in (i) of 1.5, then we find that

(6) $p \in \Sigma$ is in $\mathbf{D_P}$ iff $\forall \alpha\; \exists\, n,u (\alpha \supset u \wedge u \Vdash n \in p)$.

If however they were given as in (ii) of 1.5 by the subalgebra of global sections of $\mathbb{G}$ comprising the <u>standard numerals</u> (i.e. $\Delta n : \Delta 1 \longrightarrow \Delta\mathbb{N}$ for $n \in \mathbb{N}$), then we find that

(7) $p \in \Sigma$ is in $\mathbf{D_P}$ iff $\exists n\; \forall \alpha\; \exists\, u (\alpha \supset u \wedge u \Vdash n \in p)$.

The reader should compare (4), (5), (6) and (7) with the combination of forcing and realizability introduced by N.D. Goodman in [7]: the tripos $\mathbf{P}$ is connected to it in the same way that the effective tripos is connected to ordinary realizability. (In fact in [7] the forcing is of a more classical kind, using cofinal covers: we could make P correspond to it more closely by replacing $\Omega(\mathbb{N}^{\mathbb{N}})$ by $\Omega(\mathbb{N}^{\mathbb{N}})_{\neg\neg}$.)

To complete the round of examples we shall consider a "realizability followed by realizability" iteration. In fact we will demonstrate a nice closure property of the effective topos construction: if we carry out this construction on a realizability topos the result of the iteration is another realizability topos. To do this we need to examine the natural number object and partial recursive functions in such toposes. The basic tool is:

6.6 Recursion Theorem for combinatory algebras

Let $\mathbb{A}$ be a combinatory algebra in a topos $\mathbb{E}$. Then there is a global element $R:1 \longrightarrow A$ (namely $\lambda z.\alpha(\alpha)$, where α is $\lambda yx.z(y(y))(x)$) such that

$$\mathbb{E} \models \forall \underline{x},\underline{y} \in A(E(R\underline{x}) \wedge R\underline{x}\underline{y} \equiv \underline{x}(R\underline{x})\underline{y}).$$

$\square$

For simplicity in what follows we shall assume that $\mathbb{A}$ is a combinatory algebra in Set; however the results generalise (and we shall need the generalisation later) to any topos $\mathbb{E}$ with natural number object.

6.7 Definition

By a choice of numerals in $\mathbb{A}$ we shall mean a map $\mathbb{N} \longrightarrow A$, denoted $n \longmapsto \ulcorner n \urcorner$, for which there are elements $s,p,d \in A$ satisfying for all $n \in \mathbb{N}$ and $x,y \in A$ that

$$s(\ulcorner n \urcorner) \equiv \ulcorner n+1 \urcorner \quad \text{(successor)},$$

$$p(\ulcorner n+1 \urcorner) \equiv \ulcorner n \urcorner \quad \text{(predecessor)},$$

and

$$\left.\begin{array}{l} d\,x\,y\,\ulcorner 0 \urcorner \equiv x \\ d\,x\,y\,\ulcorner n+1 \urcorner \equiv y \end{array}\right\} \text{(definition by numeral cases)}.$$

There is always a choice of numerals in $\mathbb{A}$: for example define $\ulcorner n \urcorner$ inductively by

$$\ulcorner 0 \urcorner = e = \lambda x.x, \quad \ulcorner n+1 \urcorner = \langle \ulcorner n \urcorner, K \rangle ;$$

then we may take $s = \lambda x.\langle x, K\rangle$, $p = P_0$ and $d = \lambda xyz.z(Ke)(Ke)Kxy$.

Given a choice of numerals in $\mathbb{A}$, using d,p and R the recursion combinator of 6.6, we obtain:

6.8 Primitive Recursion for combinatory algebras

There is P in A such that for all $x,y \in A$ $EP\langle x,y\rangle$ and

$$\begin{cases} P\langle x,y\rangle\langle z,\ulcorner 0 \urcorner\rangle \equiv xz \\ P\langle x,y\rangle\langle z,\ulcorner n+1 \urcorner\rangle \equiv y\langle\langle z,\ulcorner n \urcorner\rangle, P\langle x,y\rangle\langle z,\ulcorner n \urcorner\rangle\rangle \end{cases}$$

all $n \in \mathbb{N}$, $z \in A$.

□

6.9 Definition

Say that a partial function $\varphi:\mathbb{N}^k \longrightarrow \mathbb{N}$ is weakly representable in $\mathbb{A}$ (with given numerals) iff there is an f in A such that for all $n_1,\ldots,n_k, m \in \mathbb{N}$

$$\varphi(n_1,\ldots,n_k) = m \quad \Rightarrow \quad f\langle \ulcorner n_1 \urcorner,\ldots,\ulcorner n_k \urcorner\rangle = \ulcorner m \urcorner.$$

Then since the initial functions are weakly representable, by 6.8 and a similar result for minimalization, we have:

6.10 Proposition

All the partial recursive functions $\mathbb{N}^k \longrightarrow \mathbb{N}$ are weakly representable in $\mathbb{A}$.

□

Remark

If we assume that $\mathbb{A}$ is not the degenerate algebra $\mathbb{O}$, then $n \longmapsto \ulcorner n \urcorner$ will be an injection and we can strengthen 6.10 by deleting the word "weakly" (where φ is representable iff we have both the implication in 6.9 and its converse).

Now let $R_{\mathbb{A}}$ be the realizability tripos on $\mathbb{A}$ with designated truth-values the inhabited subsets of PA (as in (i) of 1.5). The predicate $n \longmapsto \{n\}$ on $\mathbb{N}$ determines a canonical subobject of $\Delta\mathbb{N}$ in Set[$R_{\mathbb{A}}$], which we shall denote $N \rightarrowtail \Delta\mathbb{N}$. Given a function $\varphi : \mathbb{N}^k \longrightarrow \mathbb{N}$, let $R_\varphi : \mathbb{N}^k \times \mathbb{N} \longrightarrow PA$ be

$$R_\varphi(\vec{n},m) = \{ <\ulcorner n_1 \urcorner, \ldots, \ulcorner n_k \urcorner, \ulcorner m \urcorner> \mid \varphi(\vec{n}) = m \}.$$

Then R_φ is a strict, single-valued relation from N^k to N. Clearly if φ is weakly representable in $\mathbb{A}$, R_φ will also be total and so represent a function $N^k \longrightarrow N$ in Set[$R_{\mathbb{A}}$] which we shall denote by $\ulcorner\varphi\urcorner$. In particular the zero and successor functions on $\mathbb{N}$ give maps $\ulcorner 0 \urcorner : 1 \longrightarrow N$ and $\ulcorner s \urcorner : N \longrightarrow N$.

6.11 Proposition

$1 \xrightarrow{\ulcorner 0 \urcorner} N \xrightarrow{\ulcorner s \urcorner} N$ is a natural number object in Set[$R_{\mathbb{A}}$].

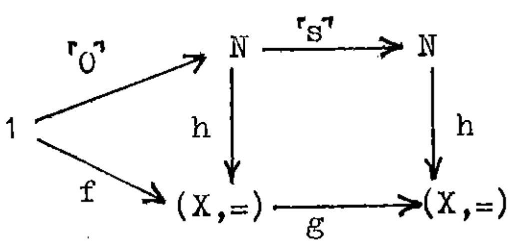
Proof

Given $1 \xrightarrow{f} (X,=) \xrightarrow{g} (X,=)$ in Set[$R_{\mathbb{A}}$], represented by $*,x \longmapsto F(x)$ and $x,x' \longmapsto G(x,x')$ say, define $H : \mathbb{N} \times X \longrightarrow PA$ inductively by

$$H(0,x) = [\![0 \in N \wedge F(x)]\!],$$

$$H(n+1,x) = [\![(n+1) \in N \wedge \exists x' (H(n,x') \wedge G(x',x))]\!]$$

for $n \in \mathbb{N}$, $x \in X$. Using 6.8 we can show that H represents a map $h : N \longrightarrow (X,=)$ in Set[$R_{\mathbb{A}}$] which makes



commute and that h is the unique such map. (For example, to see that it is single-valued choose $a, a', b \in A$ such that for all $x, x', x'' \in X$

$$a \in [\![F(x) \wedge F(x') \rightarrow x =_X x']\!],$$

$$a' \in [\![G(x'',x) \wedge G(x'',x') \rightarrow x =_X x']\!]$$

and $\quad b \in [\![x =_X x' \wedge G(x,x'') \rightarrow G(x',x'')]\!]$.

Then using 6.8 obtain $c \in A$ such that

$$\begin{cases} c<y,z> \equiv a<(y)_1,(z)_1> & \text{if } y_0 = \ulcorner 0 \urcorner, \\ c<y,z> \equiv a'<b<c<(y_1)_0,(z_1)_0>,(y_1)_1>,(z_1)_1> & \text{if } y_0 = \ulcorner n+1 \urcorner. \end{cases}$$

Hence $c \in [\![H(n,x) \wedge H(n,x') \rightarrow x =_X x']\!]$, all $n \in \mathbb{N}$, $x \in X$.)

□

6.12 Proposition

The formula (of Heyting Arithmetic) expressing partial recursive application of numbers, $\{\underline{n}\}(\underline{m}) = \underline{k}$, may be interpreted in $Set[R_{\mathbb{A}}]$ by $\| App_{\mathbb{A}} \| \rightarrowtail N \times N \times N$ where $App_{\mathbb{A}} : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \longrightarrow PA$ is the predicate

$$App_{\mathbb{A}}(n,m,k) = \{ <\ulcorner n \urcorner, \ulcorner m \urcorner, \ulcorner k \urcorner> \mid \{n\}(m) = k \}.$$

Proof

First note that if $\varphi : \mathbb{N}^k \longrightarrow \mathbb{N}$ is primitive recursive, then the corresponding primitive recursive function in $Set[R_{\mathbb{A}}]$ is precisely the map $\ulcorner \varphi \urcorner : N^k \longrightarrow N$ defined before 6.11 (this follows from the definition in 6.11 of N and the initial functions, by induction on the description of φ).

Now $\{\underline{n}\}(\underline{m}) = \underline{k}$ may be taken to be $\exists \underline{y} (T(\underline{n},\underline{m},\underline{y}) = 1 \wedge U(\underline{y}) = \underline{k})$ where Kleene's T and U are primitive recursive. This is interpreted in $Set[R_{\mathbb{A}}]$ by

$$[\![\exists y \in N (R_T(n,m,y,1) \wedge R_U(y,k)]\!]$$

which equals

$$\{ <\ulcorner y \urcorner, \ulcorner n \urcorner, \ulcorner m \urcorner, \ulcorner y \urcorner, \ulcorner 1 \urcorner, \ulcorner y \urcorner, \ulcorner k \urcorner> \mid T(n,m,y) = 1 \text{ and } U(y) = k \}$$

and this is isomorphic to the given predicate $App_{\mathbb{A}}$, since

$$n,m \longmapsto \text{the least } y \text{ such that } T(n,m,y) = 1$$

is partial recursive and hence by Proposition 6.10 is weakly representable.

□

Thus in $Set[R_{\mathbb{A}}]$ we have a combinatory algebra, namely N with partial recursive application: let P be the realizability tripos on this algebra, with designated truth-values given (as in (i) of 1.5) by the inhabited elements of PN. The hypotheses of Theorem 6.2 apply to

$$Set \xrightarrow[\Delta_{R_{\mathbb{A}}}]{} Set[R_{\mathbb{A}}] \xrightarrow[\Delta_{P}]{} Set[R_{\mathbb{A}}][P],$$

and so we have that $P\circ\Delta_{R_{\mathbb{A}}}^{op}$ is a Set-tripos with $Set[P\circ\Delta_{R_{\mathbb{A}}}^{op}]$ equivalent to $Set[R_{\mathbb{A}}][P]$.

Now since

$$P(\Delta_{R_{\mathbb{A}}} I) \cong Set[R_{\mathbb{A}}](\Delta_{R_{\mathbb{A}}} I \times N, \Omega)$$
$$\cong \{\varphi \in R_{\mathbb{A}}(I\times\mathbb{N}) \mid \varphi \text{ is a strict relation for } \Delta I\times N\},$$

we may replace $P\circ\Delta_{R_{\mathbb{A}}}^{op}$ by P', where

$$P'(I) = \{\varphi : I\times\mathbb{N} \longrightarrow PA \mid R_{\mathbb{A}} \models \forall \underline{i},\underline{n}(\varphi(\underline{i},\underline{n}) \rightarrow \underline{n} \in N)\ \}$$

(and for $f : I \longrightarrow J$, $P'f(\varphi) = \varphi\circ(f\times id)$). Let us calculate the pre-order on each $P'(I)$.

Recalling the definition of the realizability implication $\rightarrow_P : PN \times PN \longrightarrow PN$ from 1.5, we find that for $\varphi, \psi \in P'(I)$, $\varphi \rightarrow_{P'} \psi$ is the predicate

(1) $(\varphi \rightarrow_{P'} \psi)(i,n) = [\![\ n \in N \wedge \forall m(\varphi(i,m) \rightarrow \exists k(App(n,m,k) \wedge \psi(i,k)))\ ]\!]$

with App defined as in Proposition 6.12. Then $\varphi \vdash_I^{P'} \psi$ iff $\top_I \vdash_I^{P'} \varphi \rightarrow \psi$ iff $[\![\ \exists n(n \in N \wedge \forall i(\varphi \rightarrow \psi)(i,n))\ ]\!]$ is inhabited; therefore

(2) $\varphi \vdash_I^{P'} \psi$ iff there are $n \in \mathbb{N}, a \in A$ with $a \in (\varphi \rightarrow_{P'} \psi)(i,n)$, all $i \in I$.

We can now show that P' is actually equivalent to a realizability tripos:

<u>6.13 Proposition</u>

(i) There is a combinatory algebra, $e\mathbb{A}$, with underlying set $\mathbb{N} \times A$ and application given by

$$(n,a)(n',a') \equiv (n(n'), a\langle \ulcorner n' \urcorner, a' \rangle).$$

(ii) If $R_{e\mathbb{A}}$ denotes the realizability tripos on $e\mathbb{A}$ with designated truth-values the inhabited subsets of $\mathbb{N}\times A$, then $R_{e\mathbb{A}} \simeq P'$ as Set-triposes. Thus $\mathrm{Set}[R_{\mathbb{A}}][P] \simeq \mathrm{Set}[R_{e\mathbb{A}}]$.

Proof

(i) By Proposition 6.10 we can find $u \in A$ which weakly represents partial recursive application on $\mathbb{N}$, i.e.

$$n(m) = k \quad \Rightarrow \quad u\langle \ulcorner n \urcorner, \ulcorner m \urcorner\rangle = \ulcorner k \urcorner.$$

Then writing $y*z$ for $\langle u\langle (y)_0, (z)_0\rangle, (y)_1 z\rangle$, we can take the K and S combinators for $e\mathbb{A}$ to be

$$(K, \lambda x.K(x)_1) \quad \text{and} \quad (S, \lambda xyz.(x)_1 z(y*z)).$$

(ii) Given $\varphi : I \longrightarrow P(\mathbb{N}\times A)$, let $1\varphi : I\times\mathbb{N} \longrightarrow PA$ be

$$1\varphi(i,n) = \{\langle \ulcorner n \urcorner, a\rangle \mid (n,a) \in \varphi(i)\}.$$

Since the unpairing combinator P_0 is in $[\![\, 1\varphi(i,n) \to_{\mathbb{A}} n \in N \,]\!]$ all $i \in I, n \in \mathbb{N}$, we have $1\varphi \in P'(I)$.

If $\varphi' \in R_{e\mathbb{A}}(I)$ as well, and $\varphi \vdash_I \varphi'$, then there are $n \in \mathbb{N}, a \in A$ with $(n,a) \in \varphi i \to_{e\mathbb{A}} \varphi' i$, all $i \in I$. Then from the description (1) of $\to_{P'}$ given above, we have

$$\lambda x.\langle\langle \ulcorner n \urcorner, (x)_0, u\langle \ulcorner n \urcorner, (x)_0\rangle\rangle, \langle \ulcorner n \urcorner, a\rangle * x\rangle \in (1\varphi \to 1\varphi')(i,n)$$

all $i \in I$. Hence by (2) $1\varphi \vdash_I 1\varphi'$. Since the assignment $\varphi \longmapsto 1\varphi$ is clearly natural in I, we therefore have a Set-indexed functor $1 : R_{e\mathbb{A}} \longrightarrow P'$.

Furthermore 1 is full. For if $1\varphi \vdash_I 1\varphi'$, then by (1) and (2) there are $n \in \mathbb{N}$ and $a \in A$ with

$$a \in \{\ulcorner n \urcorner\} \wedge \bigcap_m \Big(1\varphi(i,m) \to \bigcup_k \big(\mathrm{App}(n,m,k) \wedge 1\varphi'(i,k)\big)\Big)$$

all $i \in I$. Hence $(a)_0 = \ulcorner n \urcorner$ and given $(m,b) \in \varphi(i)$ we have

$$(a)_1\langle \ulcorner m \urcorner, b\rangle \in \mathrm{App}(n,m,k) \wedge 1\varphi'(i,k),$$

some $k \in \mathbb{N}$: so $n(m) = k$ and $((a)_1\langle \ulcorner m \urcorner, b\rangle)_1 = \langle \ulcorner n(m) \urcorner, c\rangle$ where $(n(m), c) \in \varphi'(i)$. Therefore $(n, \lambda b.(((a)_1 b)_1)_1) \in \varphi i \to_{e\mathbb{A}} \varphi' i$, all $i \in I$, and hence $\varphi \vdash_I \varphi'$.

Given $\psi \in P'(I)$, define $r\psi : I \longrightarrow P(\mathbb{N}\times A)$ by

$$r\psi(i) = \{(n,a) \mid a \in \psi(i,n)\}.$$

Now $b \in 1r\psi(i,n)$ iff there is $a \in A$ with $b = <\ulcorner n \urcorner, a>$ and $(n,a) \in r\psi(i)$,
iff there is $a \in A$ with $b = <\ulcorner n \urcorner, a>$ and $a \in \psi(i,n)$.

So $P_1 \in [\![\forall i,n(1r\psi(i,n) \to \psi(i,n)]\!]$, and choosing an f in $[\![\forall i,n(\psi(i,n) \to n \in N)]\!]$, we have $\lambda a.< f(a),a>$ in $[\![\forall i,n(\psi(i,n) \to 1r\psi(i,n)]\!]$. Therefore $1r\psi \dashv\vdash \psi$ in $R_{\mathbb{A}}(I \times \mathbb{N})$, so they determine the same map $\Delta_{R_{\mathbb{A}}}(I) \longrightarrow PN$ in Set$[R_{\mathbb{A}}]$ and hence $1r\psi \dashv\vdash \psi$ in $P'(I)$.

It follows that $1 : R_{e\mathbb{A}} \longrightarrow P'$ is an equivalence of Set-indexed categories.

□

6.14 Remark

The results of 6.11,6.12 and 6.13 apply to realizability triposes whose designated truth-values are given as in (i) of 1.5. Let us note what happens if they are given as in (ii) of 1.5:

Suppose that $\mathbb{A}$ is a combinatory algebra in a topos E with natural number object $1 \xrightarrow{0} N_E \xrightarrow{s} N_E$, and that we have a choice of numerals $N_E \xrightarrow{\ulcorner\cdot\urcorner} A$ (and $s,p,d : 1 \longrightarrow A$) for $\mathbb{A}$. Let **A** be a subalgebra of $E(1,\mathbb{A})$, and **R** be the realizability tripos on $\mathbb{A}$ with designated truth-values given as in (ii) of 1.5 by **A**. If we suppose that $\ulcorner 0 \urcorner, s,p,d \in$ **A** (which will be the case if they are defined as after 6.7), then every partial recursive function is weakly represented by an element of **A**. Then defining $N \rightarrowtail \Delta_R N_E$ in E[**R**] as above, the equivalents of Propositions 6.11 and 6.12 go through for this case.

Let **P** denote the realizability E[**R**]-tripos on N with designated truth-values now given as in (ii) of 1.5 by the subalgebra of standard numerals $\mathbf{N} = \{s^n 0 : 1 \longrightarrow N \mid n \in \mathbb{N}\}$ (which by 5.3(ii), happen to be all the global sections of N in $\mathbb{E}[\mathbf{R}]$). Then consider $\mathbb{E}[\mathbf{R}][\mathbf{P}]$. The proof of 6.13(i) shows that if we define $e\mathbb{A}$ in $\mathbb{E}$ as before, it is a combinatory algebra with $\mathbf{N}\times\mathbf{A}$ as a subalgebra. Then just as in 6.13(ii) we can prove:

Proposition

The $\mathbb{E}$-tripos $\mathbf{P}\cdot\triangle_{\mathbf{R}}^{op}$ is equivalent to the realizability tripos $\mathbf{R}'$ on $e\mathbb{A}$ with designated truth-values given (as in (ii) of 1.5) by the subalgbra $\mathbf{N}\times\mathbf{A}$ of $\mathbb{E}(1, e\mathbb{A})$. In particular $\mathbb{E}[\mathbf{R}][\mathbf{P}] \simeq \mathbb{E}[\mathbf{R}']$, a realizability topos.

□

7. THE EFFECTIVE TOPOS CONSTRUCTION

In this final chapter we will use the results of the previous chapters to study the construction which sends a topos E, with natural number object, to the effective topos eE defined from it. We obtain two principal results:

Firstly we show that $E \longmapsto eE$ is the object part of a functor which is left adjoint to the inclusion of a subcategory into a certain category that will be defined presently. This gives a categorical characterisation of the effective topos construction, i.e. the topos eE (or more precisely the functor $\Delta: E \longrightarrow eE$ defined in Chapter 3) has a particular "universal property".

Secondly we show that the extent to which the construction $E \longmapsto eE$ fails to be idempotent is measured by a monad. The algebras for this monad are identified for the special case of realizability toposes.

To be able to state these results, we first need to examine from the point of view of combinatory algebras, the effect of applying a functor to a natural number object (NNO) in a topos E. We will denote this NNO by

$$1 \xrightarrow{\quad 0 \quad} N_E \xrightarrow{\quad s \quad} N_E.$$

7.1 Definitions

Recall that if $L: E \longrightarrow F$ is a functor between toposes (with NNO), it is called <u>regular</u> if it preserves finite limits and image factorizations and <u>exact</u> if further it preserves all finite colimits. **Reg** (respectively **Exact**) will denote the (bi)category of toposes with NNO and regular (respectively exact) functors between them. It is a result of P.J.Freyd that a regular functor $L:E \longrightarrow F$ is exact iff it preserves the NNO.

7.2 The comparison morphism λ

Suppose that $L:E \longrightarrow F$ is a functor between toposes which preserves finite limits. Then there is a comparison morphism $\lambda_L : N_F \longrightarrow L(N_E)$ in F, namely the unique morphism making

$$\begin{array}{ccccc} 1 & \xrightarrow{\;0\;} & N_F & \xrightarrow{\;s\;} & N_F \\ \cong\downarrow & & \downarrow\lambda_L & & \downarrow\lambda_L \\ L(1) & \xrightarrow[L(0)]{} & L(N_E) & \xrightarrow[L(s)]{} & L(N_E) \end{array}$$

commute. Furthermore, if $f:N^k \longrightarrow N$ is a primitive recursive function, then (by induction on the description of f) we have that

$$\begin{array}{ccc} N^k & \xrightarrow{\;f\;} & N \\ \lambda^k\downarrow & & \downarrow\lambda \\ (LN)^k & \xrightarrow[Lf]{} & LN \end{array}$$

commutes.

N_E with partial recursive application $\underline{n},\underline{m} \longmapsto \{\underline{n}\}(\underline{m})$ is a combinatory algebra in E; let $App_E \rightarrowtail (N_E)^3$ be the interpretation of the formula $\{\underline{n}\}(\underline{m}) = \underline{k}$ in E. Since this formula may be taken to be $\exists\, \underline{y}(T(\underline{n},\underline{m},\underline{y})=1 \wedge U(\underline{y})=\underline{k})$ with T and U primitive recursive, it follows from the above remark that App_F factors through $L(App_E)$ in F:

$$\begin{array}{ccc} App_F & \rightarrowtail & (N_F)^3 \\ \vdots\downarrow & & \downarrow\lambda^3 \\ L(App_E) & \rightarrowtail & (LN_E)^3. \end{array}$$

Now $L(N_E)$ with application given by $L(App_E)$ is a combinatory algebra in F: we will denote this application by

$$\underline{n}',\underline{m}' \longmapsto \{\underline{n}'\}^L(\underline{m}')$$

($\underline{n}',\underline{m}'$ variables of type $L(N_E)$). With this notation, the above commutative square gives us

(7.3) $\quad F \models \forall\, \underline{n},\underline{m},\underline{k} \in N(\{\underline{n}\}(\underline{m})=\underline{k} \rightarrow \{\lambda\underline{n}\}^L(\lambda\underline{m})=\lambda\underline{k})$.

We shall be concerned with the following strengthening of this property of λ:

7.4 Definition

Say that $L:E \longrightarrow F$ conserves (partial recursive) application iff in F we have

$$\forall \underline{n},\underline{m}\in N \quad \lambda(\{\underline{n}\}(\underline{m})) \equiv \{\lambda\underline{n}\}^L(\lambda\underline{m}).$$

(Recall the meaning of $\equiv$ and E from 1.5.) In view of (7.3), this is equivalent to saying that

$$\forall \underline{n},\underline{m}\in N \quad E\{\lambda\underline{n}\}^L(\lambda\underline{m}) \rightarrow E\{\underline{n}\}(\underline{m})$$

holds in F. Since $E\{\underline{n}\}(\underline{m})$ can be interpreted in E by

$$App_E \rightarrowtail (N_E)^2 \times N_E \xrightarrow{\pi_1} (N_E)^2,$$

we see that L conserves application iff

(7.5)
$$\begin{array}{ccccc}
App_F & \rightarrowtail & (N_F)^2 \times N_F & \xrightarrow{\pi_1} & (N_F)^2 \\
\downarrow & & & & \downarrow \lambda^2 \\
L(App_E) & \rightarrowtail & (LN_E)^2 \times LN_E & \xrightarrow[\pi_1]{} & (LN_E)^2
\end{array}$$

is a pullback square in F.

7.6 Lemma

Suppose that $L:E \longrightarrow F$ conserves application.

(i) If $n \in \mathbb{N}$ and $\begin{array}{c} W_n \xrightarrow{\alpha_n} N \\ \downarrow \\ N \end{array}$ denotes the n^{th} partial recursive function (i.e. the interpretation of $\{n\}(\underline{m})=\underline{k}$), then there is a pullback square

$$\begin{array}{ccc} W_n & \rightarrowtail & N_F \\ \downarrow & & \downarrow \lambda \\ L(W_n) & \rightarrowtail & L(N_E) \end{array}$$

in F and

$$\begin{array}{ccc} W_n & \xrightarrow{\alpha_n} & N_F \\ \downarrow & & \downarrow \lambda \\ L(W_n) & \xrightarrow[L(\alpha_n)]{} & L(N_E) \end{array}$$

commutes.

(ii) The comparison morphism $\lambda : N_F \longrightarrow L(N_E)$ is a monomorphism.

<u>Proof</u>

(i) Since $L(1 \xrightarrow{\underline{n}} N_E) = 1 \xrightarrow{n} N_F \xrightarrow{\lambda} L(N_E)$, the result follows by substituting n for $\underline{n}$ in 7.4.

(ii) Let $p : N \times N \cong N$ be a primitive recursive pairing map with primitive recursive inverse $\langle p_0, p_1 \rangle : N \cong N \times N$. Then if in (i) we take $n \in \mathbb{N}$ with

$$E\{n\}(\underline{m}) \leftrightarrow p_0\underline{m} = p_1\underline{m},$$

then $W_n \rightarrowtail N \cong N \times N$ is just the diagonal subobject Δ: hence

$$\begin{array}{ccc} N_F & \overset{\Delta}{\rightarrowtail} & N_F \times N_F \\ \lambda \downarrow & & \downarrow \lambda \times \lambda \\ L(N_E) & \overset{\Delta}{\rightarrowtail} & L(N_E) \times L(N_E) \end{array}$$

is a pullback, i.e. λ is mono.

$\square$

7.7 Lemma

Suppose $L:E \longrightarrow F$ and $K:F \longrightarrow G$ preserve finite limits. Then

(i) if L and K conserve application so does $K \circ L$;

(ii) if L and $K \circ L$ conserve application so does K;

(iii) if L is exact, it conserves application.

Proof

These follow immediately from (7.5) and the fact that $\lambda_{KL}:N_G \longrightarrow KL(N_E)$ is equal to

$$N_G \xrightarrow{\lambda_K} K(N_F) \xrightarrow{K(\lambda_L)} K(LN_E).$$

□

We shall denote by **Rca** the (non-full) subcategory of **Reg** whose morphisms are those regular functors which also conserve application. The next proposition shows that we alredy have a stock of such functors:

7.8 Proposition

Let E be a topos with NNO, $\mathbb{A}$ a combinatory algebra in E and **R** a realizability tripos on $\mathbb{A}$ (with designated truth-values given in either of the ways (i) or (ii) of 1.5). Then the "constant-objects" functor $\Delta_{\mathbf{R}}:E \longrightarrow E[\mathbf{R}]$ is a morphism in **Rca**.

Proof

By 3.4 and 3.14, $\Delta_{\mathbf{R}}$ is regular.

Given a choice of numerals in $\mathbb{A}$ (c.f. 6.7), by the analogue of Proposition 6.11 for **R**, the comparison morphism $\lambda:N \longrightarrow \Delta(N_E)$ is a monomorphism with the corresponding subobject of $\Delta(N_E)$ being canonically represented by the predicate $\underline{n} \longmapsto \{\ulcorner \underline{n} \urcorner\}$ in $\mathbf{R}(N_E)$. Similarly by 6.12, partial recursive application in

$E[\mathbf{R}]$ can be interpreted by $\|App_{\mathbb{A}}\| \rightarrowtail N^3$, where

$$App_{\mathbb{A}}(\underline{n},\underline{m},\underline{k}) = \{\langle \ulcorner \underline{n} \urcorner, \ulcorner \underline{m} \urcorner, \ulcorner \underline{k} \urcorner \rangle \mid \{\underline{n}\}(\underline{m})=\underline{k}\}.$$

Hence the formula $E\{\underline{n}\}(\underline{m})$ is interpreted in $E[\mathbf{R}]$ by $\|E_{\mathbb{A}}\| \rightarrowtail N^2$ where

$$E_{\mathbb{A}}(\underline{n},\underline{m}) = \{\langle \ulcorner \underline{n} \urcorner, \ulcorner \underline{m} \urcorner, \ulcorner \underline{k} \urcorner \rangle \mid \{\underline{n}\}(\underline{m})=\underline{k}\}.$$

Now since (by 6.10) $\underline{n},\underline{m} \longmapsto \{\underline{n}\}(\underline{m})$ is weakly representable in $\mathbb{A}$, we have

$$E_{\mathbb{A}}(\underline{n},\underline{m}) \dashv\vdash \{\underline{a} \in A \mid \exists \underline{k}\, \{\underline{n}\}(\underline{m}) = \underline{k}\} \wedge \{\, \langle \ulcorner \underline{n} \urcorner, \ulcorner \underline{m} \urcorner \rangle \,\},$$

i.e. $\quad E_{\mathbb{A}} \dashv\vdash \delta E_E \wedge [\![\, \underline{n} \in N \wedge \underline{m} \in N \,]\!]$ in $\mathbf{R}(N\times N)$,

where $E_E \rightarrowtail N_E\times N_E$ is the interpretation in E of $E\{\underline{n}\}(\underline{m})$.

Thus we have a pullback square

$$\begin{array}{ccc} \|E_{\mathbb{A}}\| & \rightarrowtail & N \times N \\ \downarrow & & \downarrow \\ \|\delta E_E\| & \rightarrowtail & \Delta(N_E \times N_E) \end{array}$$

in $E[\mathbf{R}]$. But since $N \rightarrowtail \Delta(N_E)$ is λ and $E_E \rightarrowtail (N_E)^2$ is isomorphic to $App_E \rightarrowtail (N_E)^2 \times N_E \xrightarrow{\pi_1} (N_E)^2$, we have a pullback square as in (7.5), as required.

□

<u>7.9 Definition</u>

As a particular instance of Proposition 7.8 we may take $\mathbb{A}$ to be N_E and $\mathbf{R}$ to have designated truth-values given (as in (ii) of 1.5) by the subalgebra of standard numerals $\{s^n 0 : 1 \longrightarrow N_E \mid n \in \mathbb{N}\}$. Hencefoward we will denote this E-tripos by $\mathbf{R}_E$; the topos $E[\mathbf{R}_E]$ will be denoted eE and called the (<u>external</u>) <u>effective topos</u> on E.(The internal effective topos on E, obtained by taking designated truth-values to be inhabited elements of PN_E is thus a filter-power of eE.) Thus by 7.8, the "constant-objects" functor $\Delta_E : E \longrightarrow eE$ is a morphism in **Rca**.

7.10 Lemma

Any exact functor $L: eE \longrightarrow F$ is determined up to isomorphism by the composite

$$E \xrightarrow{\ \Delta\ } eE \xrightarrow{\ L\ } F.$$

Proof

Just as in the proof of Theorem 4.8, since L preserves finite limits and colimits, it is determined by what it does to constant objects and subobjects of them. But such a subobject $\|\varphi\| \rightarrowtail \Delta(I)$ (where $\varphi: I \longrightarrow PN$ in E) is obtained by pulling back from the generic such:

$$\begin{array}{ccc}
\|\varphi\| & \rightarrowtail & \Delta(I) \\
\downarrow & \text{pb} & \downarrow{\scriptstyle \Delta\varphi} \\
\|id_{PN}\| & \rightarrowtail & \Delta(PN).
\end{array}$$

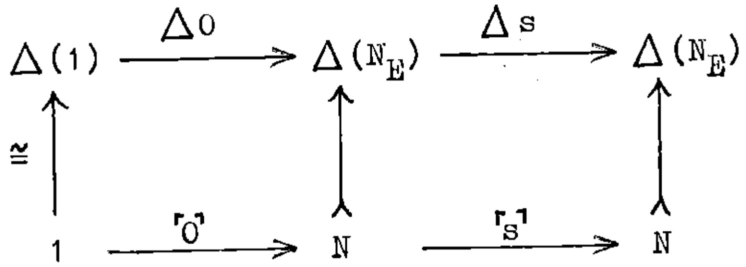
So L is determined by the restriction $L\cdot\Delta$ and by what it does to $\|id_{PN}\| \rightarrowtail \Delta(PN)$.

Now if $<\pi,\pi'>: \in_N \rightarrowtail N_E \times PN_E$ is the membership relation on N_E in E, regarding the singleton map $\{\cdot\}: N_E \longrightarrow PN_E$ as a predicate in $R_E(N_E)$ we have $id_{PN} = \exists\pi' R_E\pi(\{\cdot\})$ in $R_E(PN_E)$ (since $(\exists\pi'(\{\cdot\}\circ\pi))\underline{p} = \bigcup\{\{\pi\underline{e}\} \mid \pi'\underline{e} = \underline{p}\} = \bigcup\{\{\underline{n}\} \mid \underline{n} \in \underline{p}\} = \underline{p}$). Thus in eE, $\|id_{PN}\| \rightarrowtail \Delta(PN_E)$ is the image along $\Delta\pi'$ of the pullback of $\|\{\cdot\}\| \rightarrowtail \Delta(N_E)$ along $\Delta\pi$:

$$\begin{array}{ccccc}
 & \twoheadrightarrow & \|id_{PN}\| & \rightarrowtail & \\
\cdot & & & & \Delta(PN_E) \\
 & \rightarrowtail & \Delta(\in_N) & \xrightarrow{\ \Delta\pi'\ } & \\
\downarrow & \text{pb} & \downarrow{\scriptstyle \Delta\pi} & & \\
\|\{\cdot\}\| & \rightarrowtail & \Delta(N_E). & &
\end{array}$$

Hence L is determined by what it does to $\|\{\cdot\}\| \rightarrowtail \Delta(N_E)$ and by $L\circ\Delta$.

But just as in Proposition 6.11, we have $\|\{\cdot\}\| = N$, the natural number object in eE and

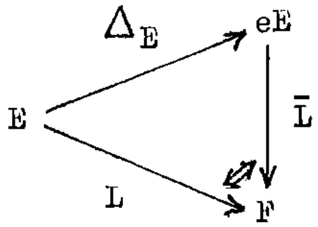


commutes. But L preserves the natural number object and hence up to isomorphism $L(N \rightarrowtail \Delta N_E)$ must be the comparison morphism $\lambda : N_F \longrightarrow L\Delta(N_E)$. Thus L is determined by $L\Delta$ as required.

□

<u>7.11 Theorem</u>

If $L:E \longrightarrow F$ is a regular functor which conserves application, then there is an exact functor $\bar{L}:eE \longrightarrow F$ such that



commutes up to isomorphism, and up to (unique) isomorphism $\bar{L}$ is the unique morphism with this property. Thus $E \longmapsto eE$ extends to a (pseudo-)functor reflecting **Rca** into **Exact**.

<u>Proof</u>

First note that if such an $\bar{L}$ exists, then by Lemma 7.5 it is unique up to isomorphism. Furthermore, to show that $\bar{L}$ does exist we just reverse the proof of that lemma: thus we first define $\bar{L}$ on subconstant objects in the only possible way and then use the presentation of objects and maps in $e\mathbb{E}$ in terms of these subconstant objects to extend $\bar{L}$ to the whole of $e\mathbb{E}$. In other words, $\bar{L}$ is the composite

$$e\mathbb{E} = \mathbb{E}[\mathbf{R}_{\mathbb{E}}] \xrightarrow{\ \bar{l}\ } \mathbb{E}[\mathbf{Sub}_{\mathbb{F}}\circ L^{op}] \xrightarrow{\ K\ } \mathbb{F},$$

where $l:\mathbf{R}_{\mathbb{E}} \longrightarrow \mathbf{Sub}_{\mathbb{E}}\circ L^{op}$ is an $\mathbb{E}$-indexed functor and the comparison functor K was defined in 3.8. This guarantees that $\bar{L}\circ\Delta_{\mathbb{E}} \cong L$. Of course $\mathbf{Sub}_{\mathbb{F}}\circ L^{op}$ need not be a **tripos**, but if we can show that l preserves $\top, \wedge$ and $\exists$, then the induced functor $\bar{l}$ will preserve finite limits and epimorphisms; since K always preserves these, we will thus have that $\bar{L} = K\circ\bar{l}$ is regular. Then to show that $\bar{L}$ is actually exact, we can just check that it preserves the NNO of $e\mathbb{E}$.

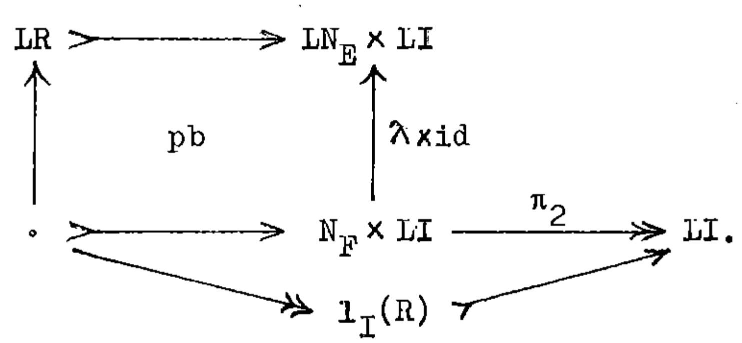
For an object I of $\mathbb{E}$ we have

$\mathbf{R}_{\mathbb{E}}(I) = \mathbb{E}(I, PN_{\mathbb{E}}) \cong \mathrm{Sub}_{\mathbb{E}}(N_{\mathbb{E}} \times I)$.

We will simplify the description of $l:\mathbf{R}_{\mathbb{E}} \longrightarrow \mathbf{Sub}_{\mathbb{E}}\circ L^{op}$ by using these bijections to identify predicates in $\mathbf{R}_{\mathbb{E}}(I)$ with subobjects of $N_{\mathbb{E}} \times I$. Under this identification the pre-order becomes

(7.12) $R \vdash_I R'$ iff there is $n \in \mathbb{N}$ with
$\mathbb{E} \models \forall \underline{m}, \underline{i}(R(\underline{m},\underline{i}) \to \exists \underline{k}(\{n\}(\underline{m}) = \underline{k} \wedge R'(\underline{k},\underline{i})))$, and in this case say that <u>n demonstrates that $R \vdash R'$</u>.

Now given $R \rightarrowtail N \times I$ in $\mathbb{E}$, define $l_I(R) \rightarrowtail LI$ in $\mathbb{F}$ as the image of a pullback:



Thus in F we have

(7.13) $\quad 1_I R(\underline{i}') \leftrightarrow \exists\, \underline{n}\; N_F\; LR(\lambda\underline{n},\underline{i}')$,

and evidently this definition makes 1_I natural in I.

We check:

(i) 1_I is order-preserving. For suppose that $n \in \mathbb{N}$ demonstrates that $R \vdash_I R'$. Applying L to (7.12) and substituting along $\lambda\times id: N \times LI \longrightarrow LN \times LI$ we thus have:

$$LR(\lambda\underline{m},\underline{i}') \rightarrow \exists\, \underline{k}'\varepsilon LN(\{Ln\}^L(\lambda\underline{m})=\underline{k}' \wedge LR'(\underline{k}',\underline{i}')).$$

Now since $L(1 \xrightarrow{n} N) \cong 1 \xrightarrow{n} N \xrightarrow{\lambda} LN$ and L conserves application, we also have:

$$\{Ln\}^L(\lambda\underline{m})=\underline{k}' \rightarrow \exists\, \underline{k}\varepsilon N\; \{n\}(\underline{m})=\underline{k} \wedge \lambda\underline{k}=\underline{k}'.$$

Combining these two gives that in F:

$$\exists\, \underline{m}\,\varepsilon N\, LR(\lambda\underline{m},\underline{i}') \rightarrow \exists\, \underline{k}\,\varepsilon N\, LR'(\lambda\underline{k},\underline{i}'),$$

i.e. that $1_I R \leqslant 1_I R'$ as subobjects of LI.

(ii) 1_I preserves $\top$. For the top element of $\mathbf{R}_E(I)$ is just the greatest subobject of $N_E \times I$, and by (7.13), 1_I sends this to the corresponding subobject of LI.

(iii) 1_I preserves $\wedge$. For the meet of $R(\underline{n},\underline{i})$ and $R'(\underline{n},\underline{i})$ in $\mathbf{R}_E(I)$ is $R(p_0\underline{n},\underline{i}) \wedge R'(p_1\underline{n},\underline{i})$, where $p_i: N \longrightarrow N$ (i=1,2) are primitive recursive unpairing functions. Thus by 7.2 and (7.13) this meet is sent by 1_I to

$$\exists\underline{k}\varepsilon N(LR(\lambda(p_0\underline{n}),\underline{i}') \wedge LR'(\lambda(p_1\underline{n}),\underline{i}')),$$

which using the pairing bijection $p: N \times N \cong N$ inverse to $\langle p_0,p_1\rangle$ is the same as

$$\exists\underline{k}\varepsilon N\; LR(\lambda\underline{k},\underline{i}') \wedge \exists\underline{m}\,\varepsilon N\; LR'(\lambda\underline{m},\underline{i}'),$$

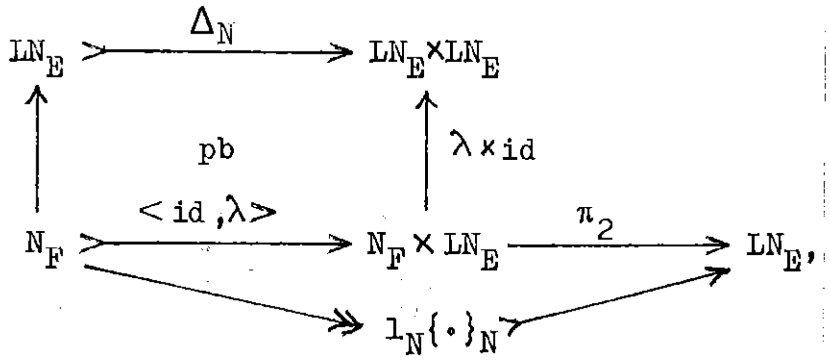
which in turn is $1_I R \wedge 1_I R'$.

(iv) 1_I preserves $\exists$. Given $f: I \longrightarrow J$ in E and $R(\underline{n},\underline{i})$ in $\mathbf{R}_E(I)$, $\exists f(R)$ is $\exists\, \underline{i}\,\varepsilon I(f\underline{i}=\underline{j} \wedge R(\underline{n},\underline{i}))$. Since L is regular we have in F that

$$1_I(\exists fR)(\underline{j}') \leftrightarrow \exists \underline{n}\in N\ \exists \underline{i}'\in LI(Lf(\underline{i}') = \underline{j}' \wedge LR(\lambda\underline{n},\underline{i}'))$$
$$\leftrightarrow \exists \underline{i}'\in LI\ (Lf(\underline{i}') = \underline{j}' \wedge 1_I R(\underline{n},\underline{i}')).$$

Thus $1_I(\exists fR) \cong \exists_{Lf}(1_I R)$ as subobjects of LJ.

We have thus shown that $\bar{I}$ and hence $\bar{L} = K\cdot\bar{I}$ are regular. It remains to prove that $\bar{L}$ preserves the NNO of eE. But the latter is given as a subconstant object $\|\{\cdot\}_N\| \rightarrowtail \Delta_E N_E$. So $\bar{L}$ sends it to $\|1_N\{\cdot\}_N\| \rightarrowtail LN_E$. Since $\{\cdot\}_N : N_E \longrightarrow PN_E$ corresponds to the diagonal subobject $\Delta_N : N_E \rightarrowtail N_E \times N_E$, we have

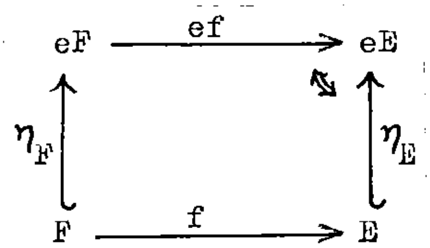


i.e. $1_N\{\cdot\}_N$ is the image of $\lambda : N_F \longrightarrow LN_E$. But by Lemma 7.6(ii) λ is a monomorphism: hence $\bar{L}$ preserves the NNO.

□

With Theorem 7.11 we thus have the promised characterisation of the effective topos construction: $\Delta : E \longrightarrow eE$ is universal amongst regular functors from E which "conserve application". In particular any such functor $L : E \longrightarrow F$ can be extended to a functor $eL : eE \longrightarrow eF$ (which is actually exact).

Now by 4.9(ii), $\Delta : E \longrightarrow eE$ is the direct image part of a geometric inclusion $\eta_E : E \hookrightarrow eE$. So given a geometric morphism $f : F \longrightarrow E$, when can we extend it to a geometric morphism $ef : eF \longrightarrow eE$ making



commute up to isomorphism? Note that since $(ef)^*: eE \longrightarrow eF$ would be exact, by Lemma 7.10 such an ef is determined by the restriction $(ef)^*\cdot(\eta_E)_*: E \longrightarrow eF$. Indeed, since f^* is exact $\Delta_F \cdot f^*: E \longrightarrow eF$ is in **Rca**, and so by Theorem 7.11 there is a unique functor $(ef)^*: eE \longrightarrow eF$ making

$$\begin{array}{ccc} & & eE \\ & \overset{\Delta_E}{\nearrow} & \\ E & & \downarrow (ef)^* \\ & \underset{\Delta_F \cdot f^*}{\searrow} \quad \cong & \\ & & eF \end{array}$$

commute. However we need some further assumption about f to ensure that this functor has a right adjoint:

7.14 Definition

Say that a geometric morphism $f: F \longrightarrow E$ between toposes with NNO conserves application iff its direct image part $f_*: F \longrightarrow E$ does so (c.f. 7.4). (By 7.7(iii), its inverse image part f^*, being exact, always conserves application.)

This condition on f_* can be stated equivalently as a condition on its left adjoint f^*. For under the isomorphism $\lambda: N_F \cong f^*(N_E)$, the comparison morphism $\lambda: N_E \longrightarrow f_*(N_F)$ is identified with the unit of the adjunction $f^* \dashv f_*$ at N, $\eta_N: N \longrightarrow f_*f^*N$. Hence f conserves application iff

$$\begin{array}{ccccc} App_E & \rightarrowtail & N^2 \times N & \xrightarrow{\pi_1} & N^2 \\ \downarrow \eta & & & & \downarrow \eta \\ f_*f^*App_E & \rightarrowtail & f_*f^*N^2 \times f_*f^*N & \xrightarrow{\pi_1} & f_*f^*N^2 \end{array}$$

is a pullback square in E; equivalently, for any pair of elements of N in E at stage I, $a,b: I \rightrightarrows N$, if $\models E\{f^*a\}(f^*b)$ at stage f^*I in F, then already $\models E\{a\}(b)$ at stage I in E.

7.15 Proposition

Any $f:F \longrightarrow E$ which conserves application extends to a geometric morphism $ef:eF \longrightarrow eE$ making

$$\begin{array}{ccc} eF & \xrightarrow{ef} & eE \\ \uparrow \eta_F & \Updownarrow & \uparrow \eta_E \\ F & \xrightarrow{f} & E \end{array}$$

commute up to isomorphism.

Proof

As we saw before 7.14, we already have a candidate for $(ef)^*$, given to us by Theorem 7.11. We can simplify its description and show it has a right adjoint by proceeding as follows:

Recall from 3.12 that there is a hyperconnected geometric morphism $h:eF = F[\mathbf{R}_F] \longrightarrow E[f_*\mathbf{R}_F]$. We will define a geometric morphism $\varphi:f_*\mathbf{R}_F \longrightarrow \mathbf{R}_E$ of E-triposes, which by Proposition 4.7 induces a geometric morphism $\bar{\varphi}:E[f_*\mathbf{R}_F] \longrightarrow E[\mathbf{R}_E] = eE$: then ef will be the composite $\bar{\varphi}\circ h$.

Just as in the proof of 7.11, we identify predicates in $\mathbf{R}_E(I)$ with subobjects $R \rightarrowtail N\times I$ in E. Similarly since

$$f_*\mathbf{R}_F(I) = E(I,f_*PN) \cong \mathrm{Sub}_F(N\times f^*I),$$

we can identify predicates in $f_*\mathbf{R}_F(I)$ with subobjects $S \rightarrowtail N\times f^*I$ in F. The pre-order on these predicates is just as in (7.12).

Then define $\varphi^*:\mathbf{R}_E(I) \longrightarrow f_*\mathbf{R}_F(I)$ by sending $R \rightarrowtail N_E\times I$ to $f^*R \rightarrowtail f^*(N_E\times I) \cong N_F\times f^*I$; and define $\varphi_*:f_*\mathbf{R}_F(I) \longrightarrow \mathbf{R}_E(I)$ by sending $S \rightarrowtail N_F\times f^*I \cong f^*(N_E\times I)$ to the pullback of f_*S along the unit of the adjunction $f^* \dashv f_*$ at $N_E\times I$:

(7.16)
$$\begin{array}{ccc} f_*S & \rightarrowtail & f_*f^*(N_E\times I) \\ \uparrow & \text{pb} & \uparrow \eta_{N\times I} \\ \varphi_*(S) & \rightarrowtail & N_E\times I, \end{array}$$

so that in E we have

$$\varphi_*S(\underline{n},\underline{i}) \leftrightarrow f_*S(\eta\underline{n},\eta\underline{i}).$$

From these definitions we get:

(i) φ^* preserves $\vdash$, T and $\wedge$. This follows directly from the fact that f^* is an exact functor which thus in particular preserves the NNO.

(ii) $\varphi^*\varphi_*S \vdash_I S$, all S in $f_*R_F(I)$. For, transposing the square in (7.16) across the adjunction $f^* \dashv f_*$ gives that

$$\begin{array}{ccc} S & \rightarrowtail & f^*(N_E\times I) \\ \uparrow & & \uparrow \text{id} \\ f^*\varphi_*S & \rightarrowtail & f^*(N_E\times I) \end{array}$$

commutes. Hence $\varphi^*\varphi_*S \leqslant S$ as subobjects of $N_F\times f^*I$, and thus (an index for) the identity partial recursive function demonstrates that $\varphi^*\varphi_*S \vdash S$.

(iii) If $\varphi^*R \vdash_I S$ then $R \vdash_I \varphi_*S$. The argument is similar to that in part (i) of the proof of Theorem 7.11. Thus if $n \in \mathbb{N}$ demonstrates that $\varphi^*R \vdash S$, i.e. we have

$$f^*R(\underline{m},\underline{i}) \rightarrow \exists \underline{k} \in N(\{n\}(\underline{m})=\underline{k} \wedge S(\underline{k},\underline{i}))$$

in F, then applying f_* and substituting along η gives

$$f_*f^*R(\eta\underline{m},\eta\underline{i}) \rightarrow \exists \underline{k}' \in f_*N(\{\eta n\}(\eta\underline{m})=\underline{k}' \wedge f_*S(\underline{k}',\eta\underline{i}))$$

in E. But since f conserves application

$$\{\eta n\}(\eta\underline{m})=\underline{k}' \rightarrow \exists \underline{k} \in N(\{n\}(\underline{m})=\underline{k} \wedge \eta\underline{k}=\underline{k}');$$

and since η is natural

$$R(\underline{m},\underline{i}) \rightarrow f_*f^*R(\eta\underline{m},\eta\underline{i}).$$

Combining these gives

$$R(\underline{m},\underline{i}) \rightarrow \exists \underline{k} \in N(\{n\}(\underline{m})=\underline{k} \wedge f_* S(\eta\underline{k},\eta\underline{i})),$$

so that n demonstrates that $R \vdash \varphi_* S$.

In view of (i) to (iii), we have that φ^*, φ_* constitute a geometric morphism of triposes $\varphi : f_* R_F \longrightarrow R_E$ as required.

It remains to check that the composites $(ef \circ \eta_F)^* \circ \Delta$ and $(\eta_E \circ f)^* \circ \Delta$ are isomorphic functors $E \longrightarrow F$. But $(ef)^*$ preserves constant objects (since both $\overline{\varphi}^*$ and h^* do) and Δ is the direct image part of the inclusion η. Hence

$$\eta_F^*(ef)^*\Delta \cong \eta_F^*(\eta_F)_* f^* \cong f^* \cong f^*\eta_E^*(\eta_E)_* \cong f^*\eta_E^*\Delta$$

and therefore $ef \circ \eta_F \cong \eta_E \circ f$.

□

The following lemma records some straightfoward consequences of Definition 7.14 (compare them with Lemma 7.7):

7.17 Lemma

Let E,F,G be toposes with NNO, and

$$G \xrightarrow{g} F \xrightarrow{f} E$$

geometric morphisms.

(i) If f and g conserve application, then so does $f \circ g$.

(ii) If $f \circ g$ conserves application, so does f.

(iii) If f is a surjection, it conserves application.

□

7.18 Definition

Lemma 7.17 implies that toposes with NNO, geometric morphisms that conserve application and natural transformations between such form a bicategory, which we shall denote by **Top**$_{ca}$.

If $f:F \longrightarrow E$ is a morphism in this bicategory, then so is the geometric morphism $ef:eF \longrightarrow eE$ constructed in 7.15: for f and η_E conserve application, hence by 7.17(i) so does $ef\circ\eta_F \cong \eta_E\circ f$, and then by 7.17(ii) ef does as well. It follows from Lemma 7.10 that e is a pseudo-functor $\mathbf{Top}_{ca} \longrightarrow \mathbf{Top}_{ca}$ (i.e. $e(id_E) \cong id_{eE}$ and $e(f\circ g) \cong ef\circ eg$, coherent isomorphisms); and by Proposition 7.15 we have that η is a natural transformation $id \longrightarrow e$.

7.19 Theorem

There is a natural transformation $\mu: ee \longrightarrow e$ which makes (e,η,μ) into a monad on $\mathbf{Top}_{ca}$, which we shall call the effective monad.

Proof

First note that by the results of Chapter 6, eeE is a realizability topos: for recall that if eN_E is the combinatory algebra in E with underlying object $N_E \times N_E$ and application

$$(\underline{n},\underline{m})(\underline{n}',\underline{m}') \equiv (\underline{n}(\underline{n}'),\underline{m}<\underline{n}',\underline{m}>),$$

then by 6.14, $eeE = eE[\mathbf{R}_E] \simeq E[\mathbf{R}']$, where $\mathbf{R}'$ is the realizability tripos on eN_E with designated truth-values given by the subalgebra $\mathbf{N}_E\times\mathbf{N}_E = \{(s^n0,s^m0) \mid n,m\in\mathbb{N}\}$.

Define maps $\mu^*:PN \longrightarrow P(N\times N)$ and $\mu_*:P(N\times N) \longrightarrow PN$ in E by

$$\mu^*(\underline{p}) = \{(\underline{n},\underline{m}) \mid \underline{n}\in\underline{p} \wedge \underline{m}\in N\},$$

and

$$\mu_*(\underline{q}) = \{<\underline{n},\underline{m}> \mid (\underline{n},\underline{m})\in\underline{q}\}.$$

One checks easily that μ^*,μ_* constitute a geometric morphism of E-triposes $\mu:\mathbf{R}' \longrightarrow \mathbf{R}_E$, inducing a geometric morphism $\bar{\mu}$ of toposes as in 4.7. Then let μ_E be

$$eeE \simeq E[\mathbf{R}'] \xrightarrow{\bar{\mu}} E[\mathbf{R}_E] = eE.$$

Since $\bar{\mu}$ preserves constant objects and since under the equivalence $eeE \simeq E[\mathbf{R}']$, $\Delta_{\mathbf{R}'}:E \longrightarrow E[\mathbf{R}']$ is identified with

$$E \xrightarrow{(\eta_E)_*} eE \xrightarrow{(\eta_{eE})_*} eeE,$$

we have that

(7.20) $\quad \mu_E^* \circ (\eta_E)_* \;\cong\; (\eta_{eE})_* \circ (\eta_E)_*$.

Hence $(\mu_E \circ \eta_{eE})^* \circ (\eta_E)_* \cong \eta_{eE}^* \circ (\eta_{eE})_* \circ (\eta_E)_* \cong (\eta_E)_* = (\mathrm{id}_{eE})^* \circ (\eta_E)_*$, so that by Lemma 7.10

$$\begin{array}{ccc} eE & \xrightarrow{\;\eta_{eE}\;} & eeE \\ & {\scriptstyle \mathrm{id}_{eE}} \searrow \;\; \cong & \downarrow {\scriptstyle \mu_E} \\ & & eE \end{array}$$

commutes up to isomorphism. In particular μ_E is a surjection and so by Lemma 7.17(iii), conserves application.

Similarly using (7.20), Lemma 7.10 and the fact that $(ef)^* \circ \eta_* \cong \eta_* \circ f^*$ for any f, it follows that μ_E is natural in E (up to isomorphism) and that

$$\begin{array}{ccc} eE & \xrightarrow{\;e(\eta_E)\;} & eeE \\ & {\scriptstyle \mathrm{id}_{eE}} \searrow \;\; \cong & \downarrow {\scriptstyle \mu_E} \\ & & eE \end{array} \qquad \text{and} \qquad \begin{array}{ccc} eeeE & \xrightarrow{\;e(\mu_E)\;} & eeE \\ {\scriptstyle \mu_{eE}} \downarrow & \cong & \downarrow {\scriptstyle \mu_E} \\ eeE & \xrightarrow[\;\mu_E\;]{} & eE \end{array}$$

commute up to (coherent) isomorphism.

□

What are the algebras for the effective monad? Note that there is no algebra structure on the topos of sets since eSet = Eff has only finite copowers of 1, so that there can be no geometric morphism θ: eSet ⟶ Set. In general a characterization of the algebras seems hard. However, if we restrict the monad to realizability toposes (of combinatory algebras in Set) we can completely characterize the algebras: we shall see that up to isomorphism, each realizability topos admits at most one algebra structure and that it does admit one when it contains Eff as a sheaf subtopos in a specially simple way.

7.21 The effective monad restricted to realizability toposes

If $\mathbb{A}$ is a combinatory algebra in Set, for simplicity we shall denote the realizability tripos on $\mathbb{A}$ (with designated truth-values the inhabited subsets of A) just by $\mathbb{A}$ again. We shall also assume that a geometric morphism $f:\mathbb{A} \longrightarrow \mathbb{B}$ between such triposes has had its inverse image functor standardized along the lines of Remark 4.10(ii), i.e. $f^*:PB \longrightarrow PA$ is given by

$$f^*(q) = \bigcup \{f(b) \mid b \in q\}$$

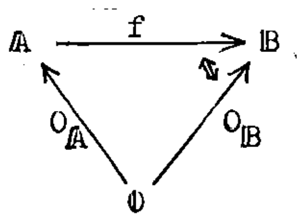
where $f:B \longrightarrow PA$ satisfies

(a) $f(b)$ is inhabited, each $b \in B$; and

(b) there is $u \in A$ such that if $b(b') = b''$ then $u \in fb \rightarrow_A (fb' \rightarrow_A fb'')$, all $b, b', b'' \in B$.

Let **RTrip** denote the bicategory of such realizability triposes and geometric morphisms (the 2-structure being given by the preorder: $f \leqslant g \Leftrightarrow f^* \vdash^{\mathbb{A}}_{PB} g^*$). Note that the degenerate combinatory algebra $\mathbb{0}$ gives an initial object for **RTrip** in the sense that for each $\mathbb{A}$ there is a geometric morphism $0_{\mathbb{A}}:\mathbb{0} \longrightarrow \mathbb{A}$, unique up to isomorphism (where $0_{\mathbb{A}}:A \longrightarrow P1$ maps each $a \in A$ to the unique $* \in 1$).

Now as in Proposition 4.7, each $f:\mathbb{A} \longrightarrow \mathbb{B}$ induces a geometric morphism of toposes $\bar{f}:\mathrm{Set}[\mathbb{A}] \longrightarrow \mathrm{Set}[\mathbb{B}]$. In particular $0_{\mathbb{A}}:\mathbb{0} \longrightarrow \mathbb{A}$ induces the inclusion $\mathrm{Set} \simeq \mathrm{Set}[\mathbb{0}] \hookrightarrow \mathrm{Set}[\mathbb{A}]$ of 5.3(ii), whose direct image part is the "constant-objects" functor. Then since



commutes up to isomorphism, so does

$$\begin{array}{ccc} \mathrm{Set}[\mathbb{A}] & \xrightarrow{\bar{f}} & \mathrm{Set}[\mathbb{B}]. \\ {\scriptstyle \bar{O}_{\mathbb{A}}}\nwarrow & & \nearrow{\scriptstyle \bar{O}_{\mathbb{B}}}\ \cong \\ & \mathrm{Set} & \end{array}$$

But by Proposition 7.8, $\bar{O}_{\mathbb{A}}$ and $\bar{O}_{\mathbb{B}}$ conserve application, and hence by Lemma 7.17 so does $\bar{f}$. Therefore we can define $e\bar{f}:\mathrm{Set}[\mathbb{A}] \longrightarrow \mathrm{Set}[\mathbb{B}]$ as in Proposition 7.15. Now by 6.13, $e\mathrm{Set}[\mathbb{A}] \simeq \mathrm{Set}[e\mathbb{A}]$ (the distinction between internal and external effective tripos disappearing in Set), and similarly for $\mathbb{B}$. Then since $e\bar{f}$ preserves constant objects, by Theorem 4.8 it is induced by a geometric morphism $ef:e\mathbb{A} \longrightarrow e\,\mathbb{B}$ of realizability triposes. We find that

$$ef:\mathbb{N}\times B \longrightarrow P(\mathbb{N}\times A)$$

is given by

$$(n,b) \longmapsto \{n\}\times f(b).$$

Thus $(ef)^*(q) = \bigcup\{ef(n,b)\mid (n,b)\in q\} = \{(n,a)\mid \exists b(a\in fb \wedge (n,b)\in q\}$,

and we may take $(ef)_*$ to be

$$(ef)_*(p) = \{(n,b)\mid b\in f_*(p_n)\},$$

where p_n is $\{a\mid (n,a)\in p\}$.

Thus $e:\mathbf{Top}_{ca} \longrightarrow \mathbf{Top}_{ca}$ restricts to a functor e from **RTrip** to itself. The monadic structure similarly restricts:

$$\eta_{\mathbb{A}}:\mathbb{A} \longrightarrow e\mathbb{A}$$

is given by

$$\eta_{\mathbb{A}}:\mathbb{N}\times A \longrightarrow PA$$
$$(n,a) \longmapsto \{<\ulcorner n\urcorner, a>\}$$

and we can take $(\eta_{\mathbb{A}})_*:PA \longrightarrow P(\mathbb{N}\times A)$ to be $p \longmapsto \mathbb{N}\times p$;

and

$$\mu_{\mathbb{A}}:ee\mathbb{A} \longrightarrow e\mathbb{A}$$

is given by

$$\mu_{\mathbb{A}}:\mathbb{N}\times A \longrightarrow P(\mathbb{N}\times\mathbb{N}\times A)$$
$$(n,a) \longmapsto \{(n,n,a)\}$$

and we can take $(\mu_{\mathbb{A}})_*:P(\mathbb{N}\times\mathbb{N}\times A) \longrightarrow P(\mathbb{N}\times A)$ to be given by sending q to $\{(n,a)\mid ((n)_0,(n)_1,a)\in q\}$. (Note that pairing in $e\mathbb{A}$ can be taken to be $<(n,a),(n',a')> = (<n,n'>,<a,a'>)$, and we can take as numerals $\ulcorner n \urcorner = (n, P_1)$, where $P_1 = \lambda a.(a)_1$ is unpairing; hence we may assume that $e\mathbb{A}$ is $\mathbb{N} \times \mathbb{N} \times A$ with application $(m,n,a)(m',n',a') = (mm', n\langle m',n'\rangle, a\langle \ulcorner\langle m',n'\rangle\urcorner, a'\rangle)$.)

7.22 Theorem

(i) If $\theta : e\mathbb{A} \longrightarrow \mathbb{A}$ is an algebra structure map for the effective monad on **RTrip**, then $\theta_* \cong \eta^*_{\mathbb{A}}$. Thus up to isomorphism each $\mathbb{A}$ admits at most one algebra structure.

(ii) The combinatory algebra $\mathbb{A}$ admits an algebra structure iff there is a map $\alpha : A \longrightarrow \mathbb{N}$ and elements $v \in A$, $u, i, r \in \mathbb{N}$ such that

(a) $a(a') = a'' \Rightarrow u(\alpha a)(\alpha a') = \alpha a''$,

(b) $\ulcorner \alpha a \urcorner = v(a)$,

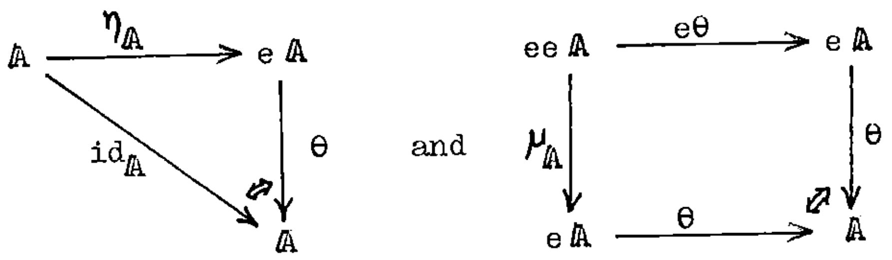
(c) $\alpha \ulcorner n \urcorner = i(n)$,

(d) $r(i(n)) = n$,

all $a, a', a'' \in A$ and $n \in \mathbb{N}$.

Proof

(i) Suppose $\theta : e\mathbb{A} \longrightarrow \mathbb{A}$ is an algebra structure map. Thus



commute up to isomorphism. In particular we have

$$\theta_* \leqslant \theta_*(\eta_{\mathbb{A}})_* \eta^*_{\mathbb{A}} \cong \eta^*_{\mathbb{A}}.$$

We must show conversely that $\eta^*_{\mathbb{A}} \leqslant \theta_*$, i.e. that $\theta^* \eta^*_{\mathbb{A}} \leqslant \mathrm{id}$.

Using the description of $\eta_{\mathbb{A}}$ given in 7.21, this means we must show that $\theta \langle \ulcorner n \urcorner, a \rangle \vdash_{(n,a)} \{(n,a)\}$. Now θ^* preserves $\wedge$, so

(1) $\theta \langle \ulcorner n \urcorner, a \rangle \dashv\vdash_{(n,a)} \theta \ulcorner n \urcorner \wedge \theta a$.

Also $\bar{\theta}^*$ preserves the natural number object of $\mathrm{Set}[\mathbb{A}]$, so θ^* preserves numerals, i.e.

(2) $\theta \ulcorner n \urcorner \dashv\vdash_n \{(n, P_1)\}$.

Finally since $\eta_{\mathbb{A}}^{*}\theta^{*} \cong \mathrm{id}$, there is $t \in A$ such that $t<\ulcorner m \urcorner, a'> = a$ whenever $(m,a') \in \theta a$; this implies that

(3) $\{(n,P_1)\} \wedge \theta a \quad \vdash_{(n,a)} \{(n,a)\}$.

So combining (1),(2) and (3) we get the required result.

(ii) First suppose that $\theta : e\mathbb{A} \longrightarrow \mathbb{A}$ is an algebra structure map. Composing with $eO_{\mathbb{A}} : \mathbb{N} \simeq e\mathbb{O} \longrightarrow e\mathbb{A}$ gives a geometric morphism $\varphi = \theta \circ eO_{\mathbb{A}} : \mathbb{N} \longrightarrow \mathbb{A}$, with $\varphi : A \longrightarrow P\mathbb{N}$ given by

$$\varphi(a) = \{m \mid \exists a' \,(m,a') \in \theta(a)\}.$$

Using (i) it is not hard to see that we may take the direct image part of φ to send $p \subseteq \mathbb{N}$ to

(4) $$\varphi_*(p) = \{\ulcorner n \urcorner \mid n \in p\}.$$

Then since $\varphi^* \dashv \varphi_*$, there is $v \in A$ with $v \in p \rightarrow_{\mathbb{A}} \varphi_*\varphi^* p$, all $p \subseteq A$. Hence if $a \in A$ then $v(a) \in \{\ulcorner n \urcorner \mid n \in \varphi(a)\}$; so we may find a map $\alpha : A \longrightarrow \mathbb{N}$ such that $v(a) = \ulcorner \alpha a \urcorner$ and $\alpha a \in \varphi(a)$, all $a \in A$. Also φ^* preserves numerals, so there is $g \in \varphi(\ulcorner n \urcorner) \rightarrow \{n\}$, all $n \in \mathbb{N}$. Then $\lambda n.g(f(n)) \in \varphi(a) \rightarrow \{\alpha a\}$, and $\alpha a \in \varphi(a)$. Therefore $\varphi(a) \dashv\vdash_a \{\alpha a\}$. Thus (a) follows from the corresponding property ((b) of 7.21) for φ; similarly (c) and (d) follow from the fact that φ^* preserves numerals; and we already have (b).

Conversely suppose we have $\alpha : A \longrightarrow \mathbb{N}$ and v,u,i,r satisfying (a) to (d). Define $\theta : A \longrightarrow P(\mathbb{N} \times A)$ by $\theta(a) = \{(\alpha a, a)\}$. Then by (a) (and since $\theta(a)$ is inhabited), $\theta^*(p) = \bigcup \{\theta(a) \mid a \in p\}$ defines a map $\theta^* : PA \longrightarrow P(\mathbb{N} \times A)$ preserving $\vdash, \top, \wedge$ and $\exists$. Using r and u we have $\theta < \ulcorner n \urcorner, a > \vdash_{(n,a)} \{(n,a)\}$, so that $\theta^* \eta_{\mathbb{A}}^* \leqslant \mathrm{id}$. Conversely $\lambda x. <v(x), x> \in \{a\} \rightarrow \{<\ulcorner \alpha a \urcorner, a>\}$, all $a \in A$, so that $\mathrm{id} \leqslant \eta_{\mathbb{A}}^* \theta^*$. Therefore $\theta^* \dashv \eta_{\mathbb{A}}^*$, and thus we have a geometric morphism $\theta : e\mathbb{A} \longrightarrow \mathbb{A}$. Since $P_1 \in \{<\ulcorner \alpha a \urcorner, a>\} \rightarrow \{a\}$, we also have $\eta_{\mathbb{A}}^* \theta^* \leqslant \mathrm{id}$, and thus $\mathrm{id} \cong \theta \circ \eta_{\mathbb{A}}$. Finally note that

$$(e\theta)^* \theta(a) = e\theta(\alpha a, a) = \{\alpha a\} \times \theta(a) = \{(\alpha a, \alpha a, a)\},$$

whilst $$\mu_{\mathbb{A}}^* \theta(a) = \mu_{\mathbb{A}}(\alpha a, a) = \{(\alpha a, \alpha a, a)\}.$$

Hence $\theta \circ e\theta \cong \theta \circ \mu_{\mathbb{A}}$. Therefore θ is indeed an algebra structure map for the effective monad.

□

7.23 Remarks

(i) It is not hard to see from the description of ef given in 7.21 that if $f: \mathbb{A} \hookrightarrow \mathbb{B}$ is an inclusion, then so is $ef: e\mathbb{A} \longrightarrow e\mathbb{B}$. In particular, applying e to $0_A: \mathbb{0} \hookrightarrow \mathbb{A}$, we get an inclusion $e0_A: \mathbb{N} \hookrightarrow e\mathbb{A}$. If the topologies on $e\mathbb{A}$ corresponding to $\eta_A: \mathbb{A} \hookrightarrow e\mathbb{A}$ and $e0_A: \mathbb{N} \hookrightarrow e\mathbb{A}$ are J and K respectively, using 5.4(i) we can show that $J+K = J_0$, the topology corresponding to $\mathbb{0} \hookrightarrow e\mathbb{A}$. Thus

$$\begin{array}{ccc} \mathbb{N} & \xhookrightarrow{e0_{\mathbb{A}}} & e\mathbb{A} \\ \uparrow & & \uparrow \eta_{\mathbb{A}} \\ \mathbb{0} & \hookrightarrow & \mathbb{A} \end{array}$$

is a pullback square in **RTrip**. (But $J_\perp < J \wedge K$.)

(ii) The proof of 7.22(ii) shows that to specify $\alpha: A \longrightarrow \mathbb{N}$ and v,u,i,r satisfying (a) to (d) is equivalent to specifying an inclusion $\varphi: \mathbb{N} \hookrightarrow \mathbb{A}$ whose direct image functor is

$$p \longmapsto \{\ulcorner n \urcorner \mid n \in p\}$$

(φ is an inclusion since $\varphi^*\varphi_*(p) = \{\alpha\ulcorner n \urcorner \mid n \in p\} \dashv\vdash_p p$, using i and r). Thus we may say: <u>$\mathbb{A}$ admits an algebra structure if $p \longmapsto \{\ulcorner n \urcorner \mid n \in p\}$ is the direct image part of an inclusion $\mathbb{N} \hookrightarrow \mathbb{A}$.</u>

7.24 Example

Let A be the set of (total) recursive functions $\mathbb{N} \longrightarrow \mathbb{N}$, ad let

$$n_0, \ldots, n_k \longmapsto \langle n_0, \ldots, n_k \rangle$$

be a primitive recursive coding of finite sequences of numbers as numbers, with primitive recursive uncoding functions $(\cdot)_k$ ($k \in \mathbb{N}$). If $f \in A$, let $\bar{f} \in A$ be $n \longmapsto \langle f0, \ldots, fn \rangle$. Using pr application on $\mathbb{N}$, we can define a partial application on A by

$$f(g) = h \quad \Leftrightarrow \quad \forall n \in \mathbb{N}(\ hn \equiv fn(\bar{g}n)\ ) \qquad (f,g,h \in A).$$

Then

$$K: n \longmapsto \lambda xy.(x)_n$$

and

$$S: n \longmapsto \lambda xyz.(x)_n z<(y)_0<(z)_0>,\ldots,(y)_n<(z)_0,\ldots,(z)_n>>$$

are "K" and "S" combinators for A, making it into a combinatory algebra, $\mathbb{A}$. We can take numerals in $\mathbb{A}$ to be the constant functions, i.e. $\ulcorner n \urcorner : m \longmapsto n$.

Now let $\alpha: A \longrightarrow \mathbb{N}$ be evaluation at 0, i.e. $\alpha(f) = f0$. If $f(g) = h$ in A, then $h0 = f0(<g0>)$; so letting $u \in \mathbb{N}$ be $\lambda xy.x<y>$, we have that u satisfies (a) of 7.10(ii). Similarly if $v \in A$ is $\lambda x.(x)_0$, then for all $f \in A$ and $n \in \mathbb{N}$ we have $vn(\bar{f}n) \equiv (\bar{f}n)_0 \equiv f0$, i.e. v(f) is defined and equals $\ulcorner f0 \urcorner = \ulcorner \alpha f \urcorner$, so that v satisfies (b). Finally since $\alpha(\ulcorner n \urcorner) = \ulcorner n \urcorner 0 = n$, we may take $i = r = e = \lambda x.x$ to satisfy (c) and (d).

Therefore $\mathbb{A}$ admits an algebra structure for the effective monad.

To complete the picture as far as the effective monad on **RTrip** is concerned, we identify the e-algebra homomorphisms:

<u>7.25 Proposition</u>

Suppose that $\mathbb{A}, \mathbb{B}$ admit e-algebra structures and that $f: \mathbb{A} \longrightarrow \mathbb{B}$ is a geometric morphism. Then f is a homomorphism of e-algebras iff there is some $b \in B$ with $b \in f_*\{\ulcorner n \urcorner | n \in p\} \rightarrow_{\mathbb{B}} \{\ulcorner n \urcorner | n \in p\}$, all $p \subseteq \mathbb{N}$.

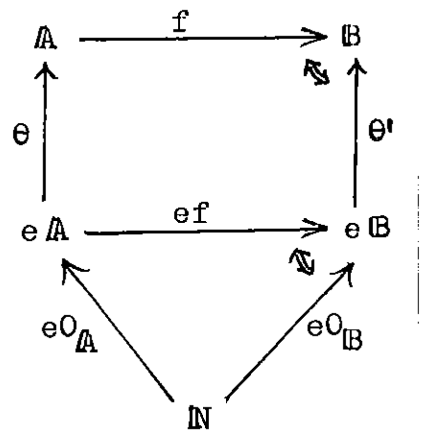
<u>Proof</u>

Let the e-algebra structure on $\mathbb{A}, \mathbb{B}$ be given, as in 7.22(ii), by maps $\alpha: A \longrightarrow \mathbb{N}$, $\beta: B \longrightarrow \mathbb{N}$, so that the structure maps are

$$\theta: e\mathbb{A} \longrightarrow \mathbb{A}, \quad \theta(a) = \{(\alpha a, a)\}$$

and $$\theta': e\mathbb{B} \longrightarrow \mathbb{B}, \quad \theta'(b) = \{(\beta b, b)\}.$$

The existence of $b \in B$ with the given property is certainly necessary for f to be a homomorphism. For if it is, then



commutes up to isomorphism. But $\theta \circ e0_{\mathbb{A}}$ and $\theta' \circ e0_{\mathbb{B}}$ are the inclusions $\varphi : \mathbb{N} \hookrightarrow \mathbb{A}$, $\varphi' : \mathbb{N} \hookrightarrow \mathbb{B}$ remarked upon in 7.23(ii), whose direct image functors are $p \longmapsto \{\ulcorner n \urcorner | n \in p\}$. Thus $f \cdot \varphi \cong \varphi'$; in particular $f_*\{\ulcorner n \urcorner | n \in p\} \vdash^{\mathbb{B}}_p \{\ulcorner n \urcorner | n \in p\}$.

Note that we always have $\{\ulcorner n \urcorner | n \in p\} \vdash^{\mathbb{B}}_p f_*\{\ulcorner n \urcorner | n \in p\}$, since this is equivalent (using the adjunction $f^* \dashv f_*$) to $f^*\{\ulcorner n \urcorner | n \in p\} \vdash^{\mathbb{A}}_p \{\ulcorner n \urcorner | n \in p\}$, and f^* preserves numerals. So if conversely there is $b \in B$ with the given property, we have $f_* \varphi_* \cong \varphi'_*$. Hence $\varphi^* f^* \cong \varphi'^*$, i.e. $\{\alpha a | a \in f(b)\} \dashv\vdash^{\mathbb{N}}_b \beta(b)$. From this it follows that $\{(\alpha a, a) | a \in f(b)\} \dashv\vdash^{e\mathbb{A}}_b \{\beta(b)\} \times f(b)$, i.e. that $\theta^* f^* \cong (ef)^*(\theta')^*$ (using the description of ef given in 7.21). Therefore f is a homomorphism.

$\square$

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