12 Quantum Computing (pm830)

The Deutsch-Jozsa algorithm distinguishes the two classes of functions $f : \{0,1\}^n \to \{0,1\}$, namely those that are constant and those that are balanced. The algorithm relies on an oracle U_f which acts on an n-qubit input register and a 1-qubit target register. Let $|x\rangle$ be the state of the input register and $|y\rangle$ be the state of the target register, such that $U_f|x\rangle|y\rangle = |x\rangle|y \oplus f(x)\rangle$.

(*a*) How should the target qubit be initialised so that the oracle U_f implements the phase map $|x\rangle \mapsto (-1)^{f(x)}|x\rangle$ on the input register? [2 marks]

(*b*) Prove that the initialisation that you proposed in Part (*a*) leads to the desired phase mapping of $|x\rangle \mapsto (-1)^{f(x)}|x\rangle$. [2 marks]

(*c*) Let $|\psi_{out}\rangle$ be the state of the input register immediately before the final computational basis $\{0,1\}$ measurement in the standard Deutsch-Jozsa circuit. Starting from the initial state $|0\rangle^{\otimes n}$, derive the mathematical expression for the probability amplitude of the basis state $|0\rangle^{\otimes n}$ in $|\psi_{out}\rangle$. Your answer should be in the form of a normalised summation involving $f(x)$. [4 marks]

(*d*) Using your result from Part (*c*), determine the probability of measuring the state $|0\rangle^{\otimes n}$ in the following two cases:

(*i*) $f(x)$ is a constant function.

(*ii*) $f(x)$ is a balanced function.

Explain how this measurement result allows the algorithm to distinguish the two classes of functions. [4 marks]

(*e*) Suppose we run the Deutsch-Jozsa algorithm with $n = 4$ input qubits and one ancilla qubit, but the function f violates the promise, i.e. it is neither constant nor balanced. Specifically, $f(x) = 1$ for 4 of the 16 possible inputs, and $f(x) = 0$ for the remaining 12. Calculate the probability that the final measurement outcome is the state $|0000\rangle$. [4 marks]

(*f*) Consider a scenario where the oracle U_f is imperfect. Instead of applying a phase factor of -1 when $f(x) = 1$, it applies a phase factor $e^{i\phi}$ for some fixed angle ϕ. That is, the effective map on the input register is $|x\rangle \mapsto e^{i \cdot f(x) \cdot \phi}|x\rangle$.

If the function f is balanced, derive an expression for the probability of measuring the all-zero state $|0\rangle^{\otimes n}$ as a function of ϕ. Verify that your result yields the expected probabilities for the standard Deutsch-Jozsa cases where $\phi = 0$ and $\phi = \pi$. [4 marks]