9 Discrete Mathematics (js2878)

(*a*) Say whether the each of the following statements is true or false, and respectively provide a proof or a counterexample justifying your claim.

(*i*) Every injective function $m\colon A \hookrightarrow B$ has a retraction, *i.e.* a function $r\colon B \to A$ such that $r \circ m = \mathsf{id}_A$. [2 marks]

(*ii*) For all sets X and Y, if $\#X \leq \#Y$ then there exists a unique injective function $f\colon X \hookrightarrow Y$. [2 marks]

(*iii*) For all sets X and Y, if there exists an injective function $f\colon X \hookrightarrow Y$, then the cardinality of X is less than or equal to the cardinality of Y. [2 marks]

(*iv*) For all sets X and Y, if there exists a surjective function $f\colon X \twoheadrightarrow Y$, then the cardinality of X is greater than or equal to the cardinality of Y. [2 marks]

(*v*) For all sets X and Y, if there exist surjective functions $f\colon X \twoheadrightarrow Y$ and $g\colon Y \twoheadrightarrow X$, then X is isomorphic to Y. [1 mark]

(*b*) Prove that any subset of a countable set is countable. [3 marks]

(*c*) Let A be a finite set, so that $\#A = n$ for some $n \in \mathbb{N}$.

(*i*) What is the cardinality of the set $\mathrm{Rel}(A, A)$ of binary relations on A? [1 mark]

(*ii*) Let $\mathrm{RxRel}(A, A)$ denote the set of *reflexive* binary relations on A. (Recall that R is reflexive when $x\ R\ x$ for all $x \in A$.) Write down explicitly a function $\Phi\colon \mathrm{Rel}(A, A) \to \mathrm{RxRel}(A, A)$ that takes a relation S on A to the *smallest* reflexive relation on A containing S. Justify your answer with proof. [3 marks]

(*iii*) What is the cardinality of the set $\mathrm{RxRel}(A, A)$ of all reflexive binary relations on A? Justify your answer with proof. [4 marks]