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# Preconditions on geometrically sensitive subdivision schemes

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## Abstract

Our objective is to create subdivision schemes with limit surfaces which are surfaces useful in engineering (spheres, cylinders, cones etc.) without resorting to special cases. The basic idea explored by Sabin et al. [1] in the curve case is that if the property that all vertices lie on an object of the required class can be preserved through the subdivision refinement, it will be preserved into the limit surface also.

The next obvious step was to try a bivariate example. We therefore identified the simplest possible scheme and implemented it. However, this misbehaved quite dramatically. This note, by doing the limit analysis (which should, in hindsight, have preceded the implementation), identifies why the misbehaviour occurred, and draws conclusions about how the problems should be avoided.

## 1 The simplest geometry sensitive bivariate scheme

Sabin et al. describe an interpolating curve scheme which preserves circles [1]. We therefore sought to find an interpolating surface scheme which preserves spheres.

We wanted to make the experiment as simple as possible. We therefore chose that the new facets should be defined by  $\sqrt{3}$  rules [2], so that each new facet joins one original vertex to new face-vertices in two adjacent triangles (Figure 1). This means that the only new vertices to be positioned are ones logically in the middles of the old facets: because the scheme is interpolating there are also new ones which are simply positioned exactly at the old ones.

These decisions meant that only triangulations had a well defined refinement, not general polyhedra, and we also chose to limit this experiment to polyhedra without boundary. This was quite acceptable for a first experiment.

Our first idea was that each new face-vertex in a triangulation should be chosen to make the estimator of mean curvature there equal to the mean of the mean curvatures at the three vertices of the face.

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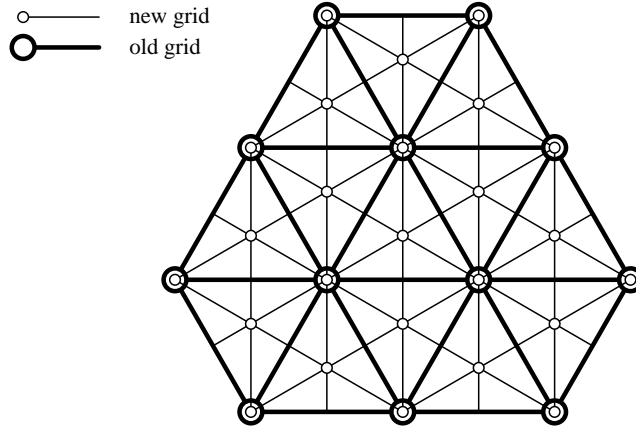


Figure 1: The  $\sqrt{3}$  refinement strategy

However, this did not seem well-enough defined, since the vertices might be of high valency and have a lot of detail in their 1-ring. We therefore used instead curvatures associated with the *edges* of the facet. Each edge has two end-vertices and two far-vertices of the facets meeting at the edge. These four vertices have a unique interpolating sphere which gives a well-defined curvature. Thus if all the vertices in the original polyhedron lay on a sphere the three curvatures would be that of the sphere and the new vertex (every new vertex) would also lie on the sphere, thus giving spherical preservation.

Because any point of a sphere gives the same curvature measure, this criterion does not completely constrain the face vertex, and so it is also necessary to position the new vertex within the sphere of the correct curvature. We chose to position it on the line perpendicular to the facet passing through the centroid of the facet. The position on that line is the place where the line is cut by the sphere segment whose curvature is the mean of the mean curvatures estimated from old vertices surrounding the facet.

The method is thus guaranteed to reproduce spheres. We now needed to investigate its behaviour when the geometry is close to spherical but not quite.

## 2 Implementation

This scheme was implemented and tried on a range of initial polyhedra. When the initial data was a regular tetrahedron, octahedron or icosahedron the rendered images looked perfect (Figures 2 and 3). Data taken less regularly over a sphere gave slight wrinkles, but for any other data significant irregularities appeared (Figure 4). Specific problems are:

- that there are significant irregularities in the spacing of the mesh. Near the left hand end of the top of the second polyhedron in Figure 4 there is a very narrow triangle. In the third polyhedron this has become a sharp corner with an inflexion. Although it is not obvious from this figure, the mesh has a self-intersection here.
- that there are ‘lumps and bumps’ at all scales, even in parts of the grid which are topologically uniform. This is particularly evident in the parts of the third polyhedron in Figure 4 which are highlighted with pink or green.

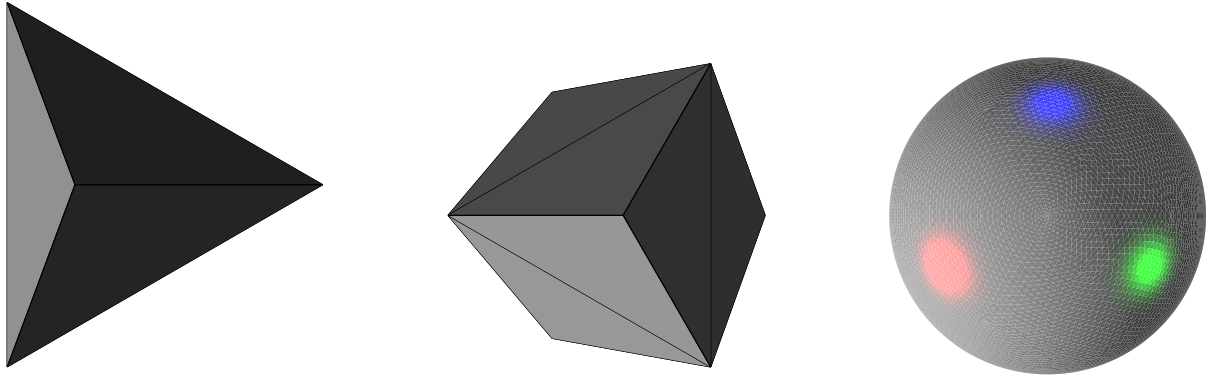


Figure 2: The refinement of a tetrahedron. Initial polyhedron left, after one refinement centre, after six refinements right. After one refinement the shape is, indeed, a triangulation of a cube.

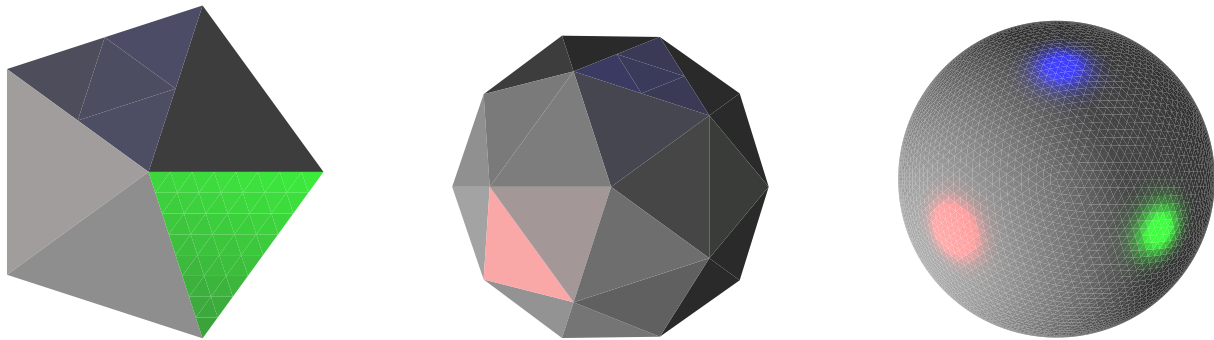


Figure 3: The refinement of an icosahedron. Initial polyhedron left, after one refinement centre, after six refinements right.

The spacing problem is particularly evident in Figure 5, in a model originally designed to illustrate a problem with Loop subdivision. Loop subdivision refines the bipyramid into a limit surface which is not convex, due to the zero saddle curvature at low valency vertices. Here the behaviour is more interesting. The spherical shape is well-maintained, but the distribution of vertices over the surface is definitely not tending to a regular mesh.

This experiment, if regarded as the building of a candidate scheme, may therefore be regarded as a failure, but some important lessons can be learned by analysing the method more deeply.

### 3 Analysis

There are two distinct ingredients of a geometry sensitive scheme of the kind we are considering, the positioning of new vertices within the surface (“tangential”) and positioning perpendicular to the surface (“normal”). Both of these need checking.

For analysis of the tangential aspect we assume that the vertices are locally coplanar and need to prove that the mesh converges to regular: for the normal analysis we assume that the vertices may be treated as functional data over a regular grid, and try to determine the Hölder continuity.

There is a circular argument here, because local convergence to planarity depends on

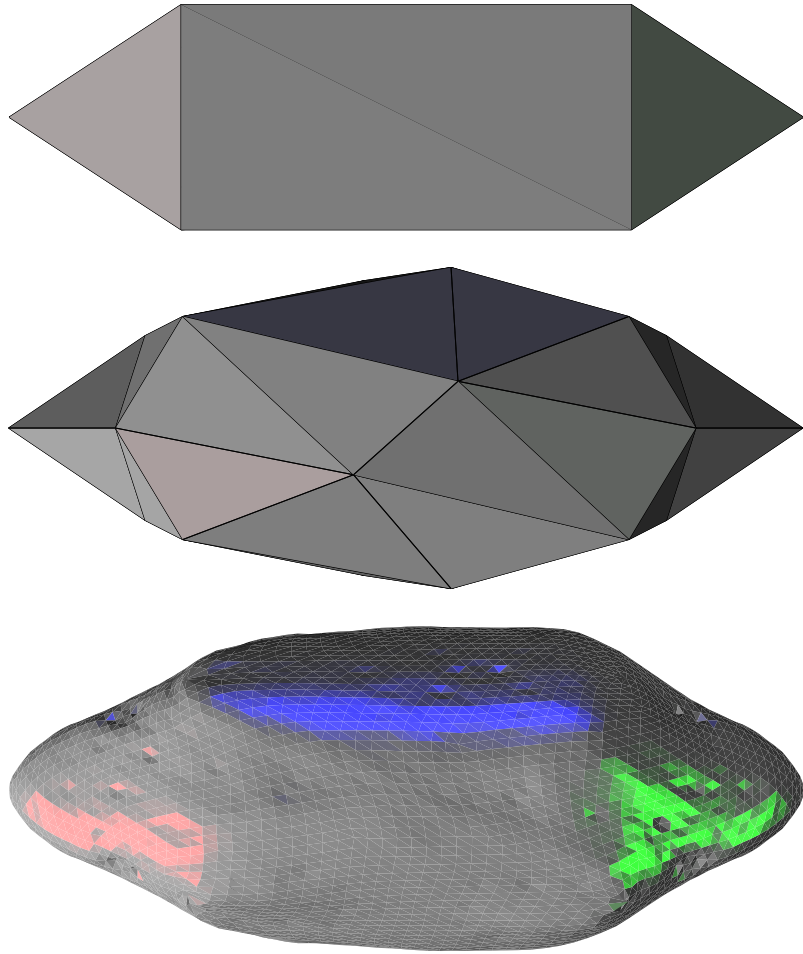


Figure 4: The refinement of a less regular figure. Initial polyhedron top, after one refinement middle, after six refinements bottom.

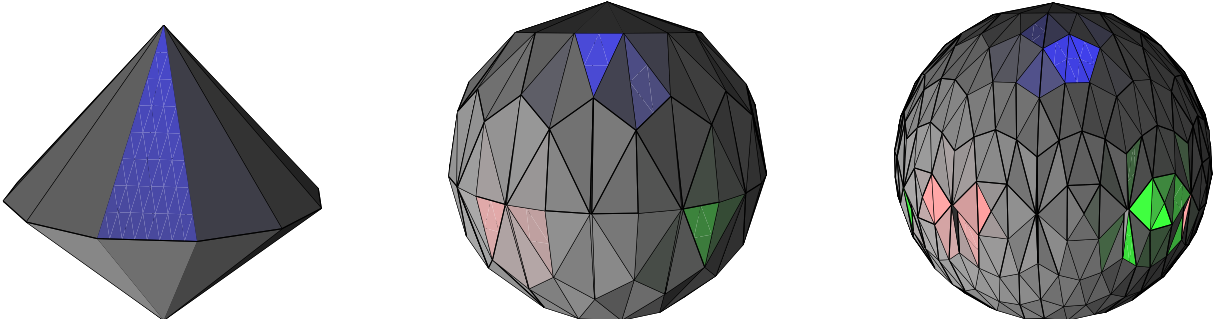


Figure 5: The refinement of an bipyramid. Initial polyhedron left, after two refinements centre, after three refinements right.

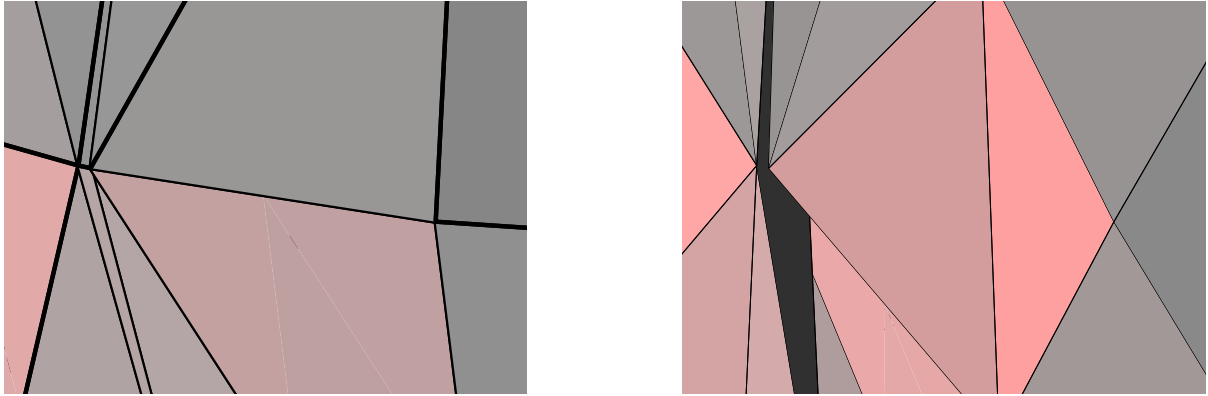


Figure 6: Zooming in on Figure 5. Two refinements left, three refinements right. The two close 6-valent vertices near the left of the two refinement image have become ‘separated’ by an edge in the three-refinement image which is no longer between them.

the normal analysis, and convergence to a regular grid depends on the tangential analysis, but if the two arguments turn out to be consistent with each other we have a plausible case, which can probably be made rigorous by showing that small departures from the assumptions make smaller changes to the conclusions drawn from them.

If either argument fails, we have identified a mechanism for bad behaviour.

### 3.1 Tangential analysis

Here we assume that the vertices are coplanar and try to prove that the mesh converges to a natural configuration which is reasonably smooth.

This is done by the usual apparatus of partitioning the scheme into separate Fourier components as described by Doo and Sabin [5] and by Kobbelt [3]. However, the interpretation of the Fourier components is rather different. The  $\omega = 0$  partition reflects the way in which the centroids of the different rings behave. The  $\omega = 1$  partition gives the natural configuration as an eigenvector, and the  $\omega = 2$  and higher partitions give distortion of the rings from affine transformations of circles. Provided that  $\mu_2 < \lambda$ , the rings tend to elliptical. Provided that  $\mu_0 < \lambda$ , the 1-ring will contain the mark point. If the eigenvector of  $\mu_0$  has the values increasing with the logical distance of the points from the centre, the rings will be correctly nested.

In the scheme being analysed, we took the new vertices to be at the intersection of the sphere of the correct curvature with the line through the centroid of the triangle in the direction of the triangle normal.

Applying this in the planar context positions each new vertex at the centroid of its triangle, which is a stationary construction. We do not have to worry about limits.

Denoting the mark point before refinement by  $A$  and after by  $a$ , and the Fourier component of the 1-ring by  $B$  before refinement and by  $b$  after, the refinement becomes

$$3 \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 1 & 2k \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix}$$

where  $k = \cos(\pi\omega/n)$  and  $n$  is the valency of the vertex whose neighbourhood we are analysing.

The matrix  $\begin{bmatrix} 3 & 0 \\ 1 & 2k \end{bmatrix}$  has only two eigenvalues, 1 and  $\frac{2}{3}$ , for the zero angular frequency, and only one,  $\frac{2k}{3}$ , for each non-zero angular frequency. The unit eigenvalue is always dominant because  $k \leq 1$  and so  $\frac{2k}{3} < 1$ .

For the particular case of regular vertices with  $n = 6$ , the eigenvalues are

$$\begin{array}{ccc} \omega & k & \frac{2k}{3} \\ 0 & 1 & \frac{2}{3} \\ \pm 1 & \frac{\sqrt{3}}{2} & \frac{1}{\sqrt{3}} \\ \pm 2 & \frac{1}{2} & \frac{1}{3} \\ 3 & 0 & 0 \end{array}$$

so that  $\lambda = \frac{1}{\sqrt{3}}$ ,  $\mu_0 = \frac{2}{3}$ ,  $\mu_1 = \frac{1}{3}$

This is the first hint of trouble. The  $\omega = 0$  component has an eigenvalue  $\mu_0$  greater than the  $\omega = 1$  eigenvalue  $\lambda$ . The displacement of the centroid of the 1-ring around a vertex from the vertex itself converges more slowly than the 1-ring shrinks. We therefore predict that unless every vertex is initially at the centroid of its 1-ring, after a finite number of steps each vertex will be outside its 1-ring. The original  $\sqrt{3}$  scheme has the same expression for its new vertices. The reason why it does not share this bad behaviour is that the new position of the old vertex moves always to lie within the 1-ring. The top left hand corner of its  $\omega = 0$  subdivision matrix is

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} / 3$$

which has eigenvalues 1 and  $\frac{1}{3}$ .

This result can also be seen by a simpler, ad-hoc argument. Consider a planar configuration in which the 1-ring is symmetrical, but the central vertex is displaced from the geometrical centre (Figure 7). The symmetry is not essential for the basic principle of the argument, but it makes the example clearer.

In Figure 7, the vertices labelled  $B$  are those of the 1-ring of vertex  $A$ , and  $C$  is their centroid. The vertices labelled  $B'$  are those of the 1-ring of  $A$  after one refinement, and their centroid is  $C'$ . We have taken the case where the  $B$  are equally spaced on a circle around  $C$ , although the substance of this argument holds more generally. Writing  $d$  for  $AC$  and  $r$  for  $BC$ , after one iteration we have that:

$$\begin{aligned} d' = AC' &= \frac{2AC}{3} = \frac{2d}{3} \\ r' = B'C' &= \frac{BC}{\sqrt{3}} = \frac{r}{\sqrt{3}} \end{aligned}$$

After two iterations

$$\begin{aligned} d'' = AC'' &= \frac{4AC}{9} = \frac{4d}{9} \\ r'' = B''C'' &= \frac{BC}{3} = \frac{3BC}{9} = \frac{3r}{9} \end{aligned}$$



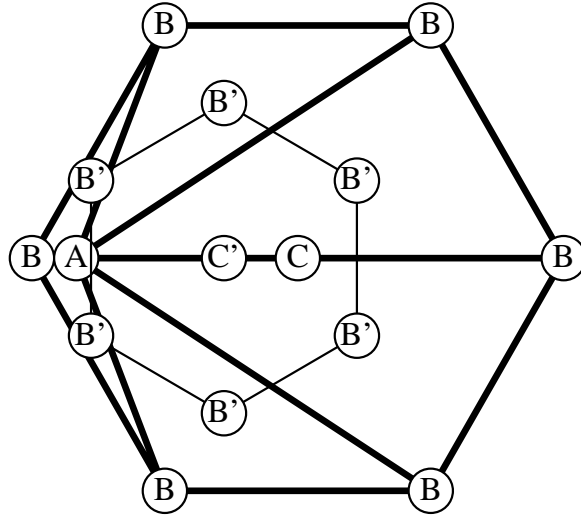


Figure 7: The behaviour of the natural configuration. After one refinement the centroid of the 1 ring moves towards the vertex by less than the shrinkage of the ring

Thus the radius of the 1-ring around its centre shrinks faster, by a ratio of  $\frac{2}{\sqrt{3}}$ , than the distance of that centre from  $A$ . If initially  $d = \rho r$ , then the number of refinement steps  $n$  before  $A$  lies outside the 1-ring is given by the relation

$$\begin{aligned}
 d^{(n)} &= r^{(n)} \\
 \left(\frac{4}{9}\right)^{n/2} d &= \left(\frac{3}{9}\right)^{n/2} r \\
 \left(\frac{4}{9}\right)^{n/2} \rho &= \left(\frac{3}{9}\right)^{n/2} \\
 \rho &= \left(\frac{3}{4}\right)^{n/2} \\
 n &= 2 \log_{3/4} \rho \\
 &= 2 \frac{\log_2 \rho}{\log_2(\frac{3}{4})} \\
 &\approx -5 \log_2 \rho
 \end{aligned}$$

If  $A$  is initially  $\frac{1}{4}$  of the way from  $C$  to the 1-ring, so that  $\rho = \frac{1}{4}$  and  $\log_2 \rho = -2$ , about 10 iterations will be needed to take it outside. This effect is therefore visible only in the seriously distorted case of Figure 4, not in the much more regular (but not completely regular after one iteration) cases of Figure 2 and 3.

The actual behaviour seen in the bipyramid example is based on this misbehaviour, but is slightly more complicated. When a vertex gets very close to its 1-ring, a facet becomes steeply inclined to the local tangent to the sphere, and the line through the centroid intersects the sphere at a relatively distant point, causing the mesh to become seriously overlapping.

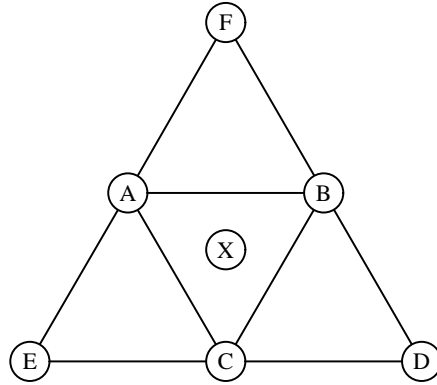


Figure 8: The configuration of points influencing new vertex  $X$ .

### 3.2 Normal analysis

Because the tangential analysis indicates serious failure modes, we cannot in fact assume that the mesh tends towards regular locally as refinement proceeds. However, it is worth continuing with this analysis to see if anything else goes wrong.

We look at the stationary limit scheme, to which the non-stationary scheme should converge in the limit, and then carry out the eigenanalysis.

The assumption (not justified in this case, but we are going to make it anyway) is that after a large number of iterations the mesh is locally almost planar; the vertices are close to a plane which we shall take as the  $xy$ -plane of a coordinate system (the distances from that plane are much less than their distances from each other) and the projections of the vertices on to that plane form a regular grid. We can then take the  $z$ -coordinates of the vertices as the numbers characterising the mesh.

We use the letters  $A, B, C, \dots, X$  both to label the points of the grid and to denote their  $z$ -coordinates. No confusion should arise.

In this context the spheres which we use as our construction criterion are closely approximated by paraboloids of revolution.

$$z = s(x^2 + y^2) + px + qy + r$$

for some coefficients  $p, q, r, s$ .

We set the  $z$ -coordinate of the new vertex  $X$  to be the mean of the  $z$ -coordinates there of three such paraboloids, one through  $ABCD$ , one through  $ABCE$  and one through  $ABCF$ .

These three numbers can each be computed simply by subtracting a linear function from all the  $z$ -coordinates, so that  $A', B'$  and  $C'$  lie in the  $xy$ -plane. The  $z$ -coordinate  $D'$  is then given by  $D - A - B + C$ .  $X'$  denotes the centroid of  $A'B'C'$ .

If we now choose the origin to lie at the centroid of  $A'B'C'$  and the scale in the  $xy$ -plane to be such that  $X'A' = X'B' = X'C' = 1$  we note that the distance  $X'D' = 2$

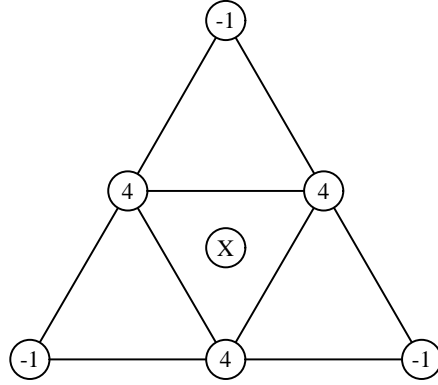


Figure 9: The stencil for the new face-vertex  $X$ . All numbers are to be divided by 9 to give a weighted mean.

The paraboloid through  $A'B'C'D'$  now intersects the  $xy$ -plane in a circle of unit radius. It becomes

$$z = s(x^2 + y^2) + c$$

and the conditions of incidence at  $A'$  for example and  $D'$  become

$$\begin{aligned} 0 &= 1^2s + c \quad \text{and} \quad D' = 2^2s + c \\ \text{Hence: } 3s &= D' \quad \text{and} \quad c = -s = -\frac{1}{3}D' \end{aligned}$$

$c$  is the height of the paraboloid over  $X'$ , which is what we need.

We then have to add back in the linear function subtracted earlier.

$$\begin{aligned} X &= \frac{-D'}{3} + \frac{A + B + C}{3} \\ &= \frac{A + B + C}{3} - \frac{D - A - B + C}{3} \\ &= \frac{2A + 2B - D}{3} \end{aligned}$$

The average of the three paraboloids is

$$\frac{1}{9}(4(A + B + C) - (D + E + F))$$

so that the stencil for the new vertices becomes that shown in Figure 9.

This scheme is a rather nice analogue of the 4-point univariate scheme. This stencil is equivalent to evaluating at  $X$  the quadratic surface through the six vertices. It is even simpler than the scheme explored by Labsik and Greiner [6]. It has been explored by Jiang and Oswald [4] who asserts that its Hölder continuity is 1.54 so that it has a well defined tangent plane everywhere, but nowhere any second derivatives (unless, of course, all the vertices originally lay on a paraboloid above a regular grid, which is the precision set).

We can also explore the level of derivative continuity at vertex points on the limit surface (including extraordinary points of valency  $n$ ) by postulating that each ring varies sinusoidally with a rotational frequency  $\omega$ . This approach, first outlined by Doo and

Sabin [5], is described in detail by Kobbelt [3]. This then leads, for each such frequency, to a relatively small linear system relating the amplitudes of the various rings after refinement to those before. There are enough of these linear systems to account for all of the components of the complete system, and so the postulate does not lead to missing anything. The matrix of each of these linear system is a block lower triangular system, and the dominant eigenvalues, apart from the unit eigenvalue coming from  $\omega = 0$  come from the second block.

$$\begin{bmatrix} 8k_1 - 2k_3 & -1 \\ 9 & 0 \end{bmatrix}$$

where  $k_j = \cos(\pi j \omega / n)$ .

The important eigenvalues are  $\mu_0$  from  $\omega = 0$ ,  $\lambda$  from  $\omega = 1$  and  $\mu_2$  from  $\omega = 2$ . and these can be tabulated against  $n$ .

$n$	$\mu_0$	$\lambda$	$\mu_2$	$\mu_0/\lambda^2$	$\mu_2/\lambda^2$
4	.333333	.600707	.333333	.923748	.923748
5	.333333	.603767	.333333	.914408	.914408
6	.333333	.577350	.333333	1.0	1.0
7	.333333	.549038	.553791	1.10579	1.837137
8	.333333	.524238	.600707	1.212893	2.18578

We therefore expect to see flat spots at low valency extraordinary points and unbounded curvature, with the saddle curvature eventually dominating, at high valency ones. Thus the performance is not catastrophic but, as might be expected from the small stencil, not particularly good either.

Note that for  $n < 6$ ,  $\mu_2$  is actually complex with the number quoted above being the modulus. For  $n = 6$ , both  $\mu_0$  and  $\mu_2$  are double eigenvalues with a Jordan block, and so the level of continuity is  $2 - \epsilon$  at the vertex points rather than 2. This is true for  $\mu_0$  for all valencies.

## 4 Conclusions

1. The fact that a subdivision construction sounds plausible is no guarantee that it will behave reasonably. It is essential that the limit scheme should be identified and analysed for both the tangential and normal aspects.
2. It is necessary for the limit schemes to behave adequately and for the actual scheme to converge sufficiently rapidly to the limit schemes. Analysis of the rate of convergence suggests that measures of curvature will probably converge fast enough towards second divided differences for the rate to be adequate [1].
3. It will probably be wise for future experiments to be designed so that the limit scheme for both the tangential and normal cases is the same, and the same as one of the well-known stationary schemes.

## 5 References

1. M. Sabin and N. Dodgson, “A circle-preserving variant of the four-point subdivision scheme”, *Mathematical Methods for Curves and Surfaces*, 275–286, Dæhlen, Mørken and Schumaker (editors), Nashboro Press 2005, ISBN 0-9728482-4-X.
2. L. Kobbelt, “ $\sqrt{3}$  subdivision”, Proc. ACM SIGGRAPH 2000, 103–112, 2000.
3. L. Kobbelt, “Using the discrete Fourier transform to analyse the convergence of subdivision schemes”, *Applied and Computational Harmonic Analysis* 4:68–91, 1998.
4. Q. Jiang and P. Oswald, “Triangular  $\sqrt{3}$ -subdivision schemes: the regular case”, <http://www.cs.umsl.edu/~jiang/webpaper/sqrt3.pdf>
5. D. Doo and M. Sabin, “Behaviour of recursive subdivision surfaces near extraordinary points”, *Computer Aided Design* 10:356–360, 1978/
6. U. Labsik and G. Greiner, “Interpolatory  $\sqrt{3}$ -subdivision”, *Computer Graphics Forum* 19(3):131–138, (Proc. Eurographics), 2000.