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Subdivision as a sequence of sampled C^p surfaces and conditions for tuning schemes

Cédric Gérot* Loïc Barthe† Neil A. Dodgson‡
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Abstract

We deal with practical conditions for tuning a subdivision scheme in order to control its artifacts in the vicinity of a mark point. To do so, we look for good behaviour of the limit vertices rather than good mathematical properties of the limit surface. The good behaviour of the limit vertices is characterised with the definition of C^2-convergence of a scheme. We propose necessary explicit conditions for C^2-convergence of a scheme in the vicinity of any mark point being a vertex of valency n or the centre of an n-sided face with n greater or equal to three.

These necessary conditions concern the eigenvalues and eigenvectors of subdivision matrices in the frequency domain. The components of these matrices may be complex. If we could guarantee that they were real, this would simplify numerical analysis of the eigenstructure of the matrices, especially in the context of scheme tuning where we manipulate symbolic terms. In this paper we show that an appropriate choice of the parameter space combined with a substitution of vertices lets us transform these matrices into pure real ones. The substitution consists in replacing some vertices by linear combinations of themselves.

Finally, we explain how to derive conditions on the eigenelements of the real matrices which are necessary for the C^2-convergence of the scheme.

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Contents

1 Introduction

A bivariate subdivision scheme defines a sequence of polygonal meshes each of whose vertices is a linear combination of vertices belonging to the previous mesh in the sequence. At the 2002 Curves and Surfaces Conference in Saint Malo, Malcolm Sabin gave to the community a few challenges about subdivision schemes. One of them was to control the artifacts that schemes could create. In [15], Sabin and Barthe catalogue some possible artefacts. Some schemes (Loop [8], Catmull-Clark [4], Doo-Sabin [5],. . .) are defined so that each polygonal mesh is the control polyhedron of a Box-Spline surface which is the limit surface of the sequence. In this case the behaviour of the limit surface is known, except around *extraordinary vertices* or *faces*. An extraordinary vertex is a vertex of the mesh whose valency is not equal to six if the mesh faces are triangles, or not equal to four if the mesh faces are quadrilaterals. An extraordinary face is a non-triangular face in a triangular lattice or a non-quadrilateral face in a quadrilateral lattice.

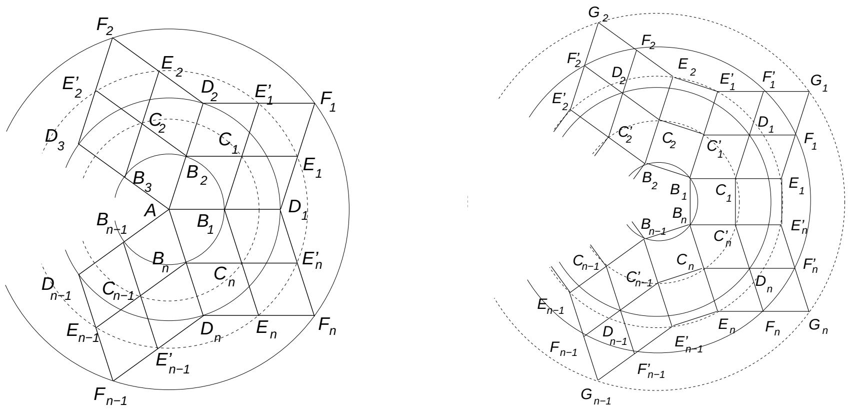
This article deals with practical conditions for tuning a scheme in order to control its artifacts in the vicinity of a *mark point*. A mark point is a point of a mesh whose vicinity keeps the same topology throughout subdivision. For some schemes, sometimes called primal, the mark point is a vertex, for others, called dual, the mark point is a face centre. In the latter case, we will refer to this face as a *mark face*. In most cases, the coefficients of the linear combinations depend only on the local topology of the mesh, and not on its geometry. Moreover, we assume the scheme to be stationary: the coefficients remain the same through the sequence of polygonal meshes.

Sabin has reviewed the state-of-the-art in tuning subdivision schemes [14]. Most work alters local coefficient in order to fulfil sufficient conditions for getting a continuous limit surface around the mark point [10, 17]. But looking for a mathematical C^2-continuity of the limit surface is possibly not the best way for controlling artifacts. For instance, Prautzsch and Umlauf tuned the Loop and Butterfly schemes in order to make them C^1- and C^2-continuous around an extraordinary vertex by creating a flat spot [11]; and a flat spot may be considered as an artifact. Furthermore, the necessary and sufficient conditions for C^2-continuity of the limit surface are not explicit if the scheme is not based on a Box-Spline.

In contrast, we may look for good behaviour of the limit vertices rather than good mathematical properties of the limit surface. In this paper, we characterise good behaviour of the limit vertices with the definition of C^2-convergence of a scheme. This definition is based on the interpretation proposed implicitly by Doo and Sabin [5]. Each control mesh is viewed not as the control polyhedron of a Box-Spline surface but as the sampling of a continuous surface. Thus the sequence of meshes are samplings of a sequence of continuous surfaces which converges uniformly towards the limit surface. Naturally, C^2-convergence of a scheme is related to the C^2-continuity of the limit surface: it is a sufficient condition for it. And because the definition of C^2-convergence of a scheme is theoretical and formal, we propose in this paper explicit but only necessary conditions. In [6], we applied these conditions at a mark point being a vertex. In this paper, these conditions are applied at any mark point being a vertex of valency n or the centre of an n-sided face with n greater or equal to three. They have already been proposed by Sabin as a condition related to the C^2-continuity of the limit surface [13]. By relating them to C^2-convergence of a scheme, we give some insights on why these conditions can be used for tuning a scheme, as has been done by Barthe and Kobbelt [3].

The necessary conditions for C^2-convergence of a scheme, proposed in this paper, concern the eigenvalues and eigenvectors of subdivision matrices in the frequency domain. Subdivision matrices in the frequency domain give the relationship between rotational frequencies coming from the discrete Fourier transform of the vertices around the mark point. That means that the components of these matrices may be complex. If we could guarantee that they were real, this would simplify numerical analysis of the eigenstructure of the matrices, especially in the context of scheme tuning where we manipulate symbolic terms. In this paper we show that an appropriate choice of the parameter space combined with a substitution of vertices let us transform these matrices into pure real ones. The substitution consists in replacing subsets of vertices by linear combinations of themselves. Of course a combination of surface samples does not, in general, belong to the same surface. Thus, the conditions given above cannot be applied directly on the new pure real matrices. In this paper, we explain how to derive conditions on the eigenelements of the real matrices which are necessary for the C^2-convergence of the scheme.

In the following section, we present the theoretical tools which we then use in Sect. 3 to establish the necessary conditions for a scheme to C^2-converge. In Sect. 4 we show how to derive real subdivision matrices in the frequency domain and conditions on their eigencomponents which are necessary to achieve the C^2-convergence of the scheme.



(a) The mark point is a vertex

(b) The mark point is a face centre

Figure 1: Labelling of the vicinity of a mark point

2 Theoretical Tools

We first present our notation and then we introduce two tools: eigenanalysis of the Fourier transformed subdivision matrices and C^p-convergence of a scheme. The eigenanalysis of the Fourier transformed subdivision matrices gives us a description of the frequencies in the limit surface around a mark point; we introduce invariances of a scheme which allow us to write these frequencies more simply. The definition of C^p-convergence allows us to derive a description of the limit points. Finally, we stress the fact that these interpretations are valid if the vertices make up a good sampling of the surfaces.

2.1 Notation

If the mark point is a vertex, let it be A and n its valency (number of outgoing edges from A). Otherwise, n is the number of edges of the mark face. We assume that the vicinity of A is made up of ordinary vertices. This hypothesis is relevant because after a subdivision step, the vertices of the mesh map to vertices with the same valency, and new vertices are created which are all ordinary. Thus, after several subdivision steps, every extraordinary vertex is surrounded by a sea of ordinary vertices. As a consequence, the vicinity of A, or of a mark face, may be divided into n topologically equivalent sectors. In the jth sector, let B_j, C_j, D_j... be an infinite number of vertices sorted from the topologically nearest vertex from the mark point to the farthest. If there exist two vertices in one sector on the same ring which are in complementary positions then they are labelled with the same letter, but with a prime put on the vertex which is further anticlockwise from the positive x-axis. An example is E and E' in Fig. 1(a). However, if the points are not in complementary positions, then they are given distinct letters.

Let $A^{(k)}$ be the mark point if it is a vertex and

$$\left\{B_j^{(k)}, C_j^{(k)}, D_j^{(k)} \dots\right\}_{j \in 1 \dots n}$$

its vicinity after k subdivision steps. All these vertices are put into an infinite vector

$$\mathbf{P^{(k)}} := \left[A^{(k)} B_1^{(k)} \cdots B_n^{(k)} C_1^{(k)} \cdots C_n^{(k)} D_1^{(k)} \cdots D_n^{(k)} \cdots\right]^T$$

if the mark point is a vertex and

$$\mathbf{P^{(k)}} := \left[B_1^{(k)} \cdots B_n^{(k)} C_1^{(k)} \cdots C_n^{(k)} D_1^{(k)} \cdots D_n^{(k)} \cdots\right]^T$$

otherwise.

Finally, a surface is C^p-smooth if there exists a C^p-diffeomorphic parametrisation of it from a subset of $\mathbb{R}^2$. We define a parametrisation domain by projecting onto $\mathbb{R}^2$ the polygonal mesh around the mark point. Depending on the injective 2D map which interpolates the vertices, the projected polygonal faces may overlap. But we are looking for surfaces which are as smooth as possible. So, we ask the polygonal mesh projection to be injective. $A^{(k)}$ is projected onto $(0,0)$ if the mark point is a vertex, and $\forall X \in \{B, C, D, \ldots\}$, $\forall j \in \{1, \ldots, n\}$, $X_j^{(k)}$ is projected onto $(x_j^{(k)}, y_j^{(k)})$. For simplicity, we ask $(x_j^{(k)}, y_j^{(k)})$ to lie on the same circle for given k and X, and to be equally distributed on the circle for given k and X, with a possible shift $d\alpha := \alpha_k - \alpha_{k-1}$ between $k-1$ and k which is the same for all X and all k:

$$(x_j^{(k)}, y_j^{(k)}) := (\varrho_X^{(k)} \cos(\theta_{(X,j,k)}), \varrho_X^{(k)} \sin(\theta_{(X,j,k)}) ,$$

where

$$\theta_{(X,j,k)} := \frac{2\pi}{n}(j + \alpha_X + \alpha_k) , \quad \alpha_k = k d\alpha . \tag{1}$$

Furthermore, because the vertices $X_j^{(k)}$ converge to the limit mark point if the scheme converges [12], we ask that $\lim_{k\to\infty}(\varrho_X^{(k)}) = 0$. The choice of the phases α_X, α_k and the radii $\varrho_X^{(k)}$ seem to remain free provided that the map is injective and the radii converge to zero. But the characterisation of C^1-convergence in Sect. 3.2 will fix the value of the phases α_X and α_k and will reduce the possible values of the radii $\varrho_X^{(k)}$.

Finally, note that for a complex number, c, we use the standard notation, $|c|$, for its modulus and use the notation φ_c for its phase.

2.2 Eigenanalysis of the Transformed Subdivision Matrices

Consideration of the relationship between the spatial and frequency domains allows us to produce necessary conditions for C^2-convergence. As mentioned in the introduction, the vertices after one step of subdivision are defined as linear combinations of the vertices in the previous mesh. As a consequence, there exists a matrix M such that

$$\mathbf{P^{(k)}} = \mathrm{M}\mathbf{P^{(k-1)}} .$$

We will refer to $\mathrm{M} = (M_{l,h})$ as the subdivision matrix. In this section we introduce the necessary notation for the definition of *transformed subdivision matrices*.

We may write the discrete rotational frequencies $\tilde{X}^{(k)}(\omega)$ of each set of vertices $\{X_j^{(k)}\}_{j\in 1\ldots n}$ by applying a shifted Discrete Fourier Transform:

$$\mathrm{DFT}\left(\left\{X_j^{(k)}\right\}\right)(\omega) = \tilde{X}^{(k)}(\omega) = \sum_{j=1}^{n} X_j^{(k)} \exp\left(-\frac{2\mathrm{i}\pi\omega}{n}(j + \phi_X + \phi_k)\right)$$

with

$$X_j^{(k)} = \frac{1}{n}\sum_{\omega=0}^{n-1} \tilde{X}^{(k)}(\omega)\exp\left(\frac{2\mathrm{i}\pi\omega}{n}(j+\phi_X+\phi_k)\right),$$

and $\mathrm{i} = \sqrt{-1}$. Usually, the discrete Fourier transform is defined without the phase $\phi_X + \phi_k$. We emphasise that a shift does not change the frequency content of the set of points, but this shift will enable us to get pure real components of subdivision matrices in the frequency domain in Sect. 4. Note that we use phases ϕ_X and ϕ_k independently of the phases α_X and α_k of the parametric space introduced in Sect. 2.1. Indeed, ϕ_X and ϕ_k will be chosen with regard to the algebraic structure of the subdivision matrix in Sect. 4.1, whereas α_X and α_k will be fixed by the characterisation of C^1-convergence in Sect. 3.2.

We now introduce two results directly related to this definition of the discrete Fourier transform and which will be used in Sect. 3 to derive necessary conditions for C^2-convergence.

Lemma 2.1 *Consider a set of vertices defined as*

$$Y_j = a\cos(\frac{2\pi\Omega}{n}(j+\alpha)) + b\sin(\frac{2\pi\Omega}{n}(j+\alpha)),$$

where a and b are constant and $\Omega \in \{0,1,2\}$. Then, letting $\delta_{\omega,\Omega}$ be the Kronecker function:

$$\delta_{\omega,\Omega} = \begin{cases} 1 & \text{if } \omega = \Omega, \\ 0 & \text{else.} \end{cases}$$

with a given $\phi_X + \phi_k$,

$$\begin{aligned} \tilde{Y}(\omega) &= \sum_{j=1}^{n} Y_j \exp\left(-\frac{2\mathrm{i}\pi\omega}{n}(j+\phi_X+\phi_k)\right) \\ &= \left(n\frac{a-ib}{2}\,\delta_{\omega,\Omega} + n\frac{a+ib}{2}\,\delta_{\omega,-\Omega}\right)\exp\left(-\frac{2\mathrm{i}\pi\omega}{n}(\alpha-\phi_X-\phi_k)\right) \end{aligned}$$

Proof

$$\begin{aligned} Y_j &= \frac{a}{2}\left[\exp\left(\frac{2\mathrm{i}\pi\Omega}{n}(j+\alpha)\right) + \exp\left(-\frac{2\mathrm{i}\pi\Omega}{n}(j+\alpha)\right)\right] \\ &\quad + \frac{b}{2i}\left[\exp\left(\frac{2\mathrm{i}\pi\Omega}{n}(j+\alpha)\right) - \exp\left(-\frac{2\mathrm{i}\pi\Omega}{n}(j+\alpha)\right)\right] \\ &= \frac{a-ib}{2}\exp\left(\frac{2\mathrm{i}\pi\Omega}{n}(j+\alpha)\right) + \frac{a+ib}{2}\exp\left(-\frac{2\mathrm{i}\pi\Omega}{n}(j+\alpha)\right) \\ \tilde{Y}(\omega) &= \sum_{j=1}^{n} Y_j \exp\left(-\frac{2\mathrm{i}\pi\omega}{n}(j+\phi_X+\phi_k)\right) \\ &= \left(\sum_{j=1}^{n} Y_j \exp\left(-\frac{2\mathrm{i}\pi\omega}{n}(j+\alpha)\right)\right)\exp\left(\frac{2\mathrm{i}\pi\omega}{n}(\alpha-\phi_X-\phi_k)\right) \end{aligned}$$

$$
\begin{aligned}
&= \left[\frac{a-\mathrm{i}b}{2}\exp\left(\frac{2\mathrm{i}\pi\alpha}{n}(\Omega-\omega)\right)\sum_{j=1}^{n}\exp\left(\frac{2\mathrm{i}\pi j}{n}(\Omega-\omega)\right)\right.\\
&\quad\left.+\frac{a+\mathrm{i}b}{2}\exp\left(-\frac{2\mathrm{i}\pi\alpha}{n}(\Omega-\omega)\right)\sum_{j=1}^{n}\exp\left(-\frac{2\mathrm{i}\pi j}{n}(\Omega+\omega)\right)\right]\exp\left(\frac{2\mathrm{i}\pi\omega}{n}(\alpha-\phi_X-\phi_k)\right)\\
&= \left[\frac{a-\mathrm{i}b}{2}\exp\left(\frac{2\mathrm{i}\pi\alpha}{n}(\Omega-\omega)\right)\frac{1-\exp\left(2\mathrm{i}\pi(\Omega-\omega)\right)}{1-\exp\left(\frac{2\mathrm{i}\pi}{n}(\Omega-\omega)\right)}\right.\\
&\quad\left.+\frac{a+\mathrm{i}b}{2}\exp\left(-\frac{2\mathrm{i}\pi\alpha}{n}(\Omega-\omega)\right)\frac{1-\exp\left(2\mathrm{i}\pi(\Omega+\omega)\right)}{1-\exp\left(\frac{2\mathrm{i}\pi}{n}(\Omega+\omega)\right)}\right]\exp\left(\frac{2\mathrm{i}\pi\omega}{n}(\alpha-\phi_X-\phi_k)\right)\\
&= \left(n\frac{a-\mathrm{i}b}{2}\delta_{\omega,\Omega}+n\frac{a+\mathrm{i}b}{2}\delta_{\omega,-\Omega}\right)\exp\left(\frac{2\mathrm{i}\pi\omega}{n}(\alpha-\phi_X-\phi_k)\right)
\end{aligned}
$$

∎

Lemma 2.2 *If for all* $j\in\{1,\ldots,n\}$, $\lim_{k\to\infty}X_j^{(k)}=0$ *then for every* $\omega\,\{1,\ldots,n\}$, $\lim_{k\to\infty}\tilde{X}^{(k)}(\omega)=0$.

Proof $\lim_{k\to\infty}\tilde{X}^{(k)}(\omega)=0$ if and only if $\lim_{k\to\infty}\left|\tilde{X}^{(k)}(\omega)\right|=0$. Furthermore,

$$
\begin{aligned}
\lim_{k\to\infty}\left|\tilde{X}^{(k)}(\omega)\right| &= \lim_{k\to\infty}\left|\sum_{j=1}^{n}X_j^{(k)}\exp\left(-\frac{2\mathrm{i}\pi\omega}{n}(j+\phi_X+\phi_k)\right)\right|\\
&= \lim_{k\to\infty}\left|\sum_{j=1}^{n}X_j^{(k)}\exp\left(-\frac{2\mathrm{i}\pi\omega}{n}j\right)\right|\left|\exp\left(-\frac{2\mathrm{i}\pi\omega}{n}(\phi_X+\phi_k)\right)\right|\\
&= \lim_{k\to\infty}\left|\sum_{j=1}^{n}X_j^{(k)}\exp\left(-\frac{2\mathrm{i}\pi\omega}{n}j\right)\right|
\end{aligned}
$$

which leads to the result. ∎

If the discrete Fourier transform is defined without the phase, it is well-known [1] that there exist transformed subdivision matrices $\tilde{\mathrm{M}}(\omega)$ such that for all ω in $\left\{-\lfloor\frac{n-1}{2}\rfloor,\ldots,\lfloor\frac{n}{2}\rfloor\right\}$,

$$\tilde{\mathbf{P}}^{(\mathbf{k+1})}(\omega)=\tilde{\mathrm{M}}(\omega)\tilde{\mathbf{P}}^{(\mathbf{k})}(\omega)$$

where, if $\omega\neq 0$,

$$\tilde{\mathbf{P}}^{(\mathbf{k})}(\omega):=\left[\tilde{B}^{(k)}(\omega)\tilde{C}^{(k)}(\omega)\tilde{D}^{(k)}(\omega)\cdots\right]^T$$

and otherwise

$$\tilde{\mathbf{P}}^{(\mathbf{k})}(0):=\left[\tilde{A}^{(k)}(0)\tilde{B}^{(k)}(0)\tilde{C}^{(k)}(0)\tilde{D}^{(k)}(0)\cdots\right]^T.$$

With our definition of the discrete Fourier transform with a phase α_k, it is clear that there exist such matrices but that they could depend on k. But because we have asked $d\alpha=\alpha_k-\alpha_{k-1}$ to be independent of k, the matrices $\tilde{\mathrm{M}}(\omega)$ do not depend on k.

For every discrete rotational frequency ω, the matrix $\tilde{\mathrm{M}}(\omega)$ is assumed to be non-defective (otherwise we should use the canonical Jordan form).

$$\tilde{\mathrm{M}}(\omega) = \tilde{\mathrm{V}}(\omega)^{-1}\tilde{\Lambda}(\omega)\tilde{\mathrm{V}}(\omega)$$

where the columns $\tilde{\mathbf{v}}_{\mathbf{l}}(\omega)$ of $\tilde{\mathrm{V}}(\omega)^{-1}$ are the right eigenvectors of $\tilde{\mathrm{M}}(\omega)$, the rows $\tilde{\mathbf{u}}_{\mathbf{l}}^{\mathbf{T}}(\omega)$ of $\tilde{\mathrm{V}}(\omega)$ are the left eigenvectors of $\tilde{\mathrm{M}}(\omega)$, and $\tilde{\Lambda}(\omega)$ is diagonal whose diagonal components $\tilde{\lambda}_l(\omega)$ are the eigenvalues of $\tilde{\mathrm{M}}(\omega)$, with $l \geq 1$. Let $L_l^-(\omega)$, $L_l(\omega)$, and $L_l^+(\omega)$ be sets of indices such that:

$$\begin{array}{llll} \text{if} & q \in L_l^-(\omega) & \text{then} & \left|\tilde{\lambda}_q(\omega)\right| < \left|\tilde{\lambda}_l(\omega)\right| \\ \text{if} & q \in L_l(\omega) & \text{then} & \left|\tilde{\lambda}_q(\omega)\right| = \left|\tilde{\lambda}_l(\omega)\right| \\ \text{if} & q \in L_l^+(\omega) & \text{then} & \left|\tilde{\lambda}_q(\omega)\right| > \left|\tilde{\lambda}_l(\omega)\right|. \end{array}$$

Finally we define $\mathcal{P}(q,\omega) := \tilde{\mathbf{u}}_{\mathbf{q}}(\omega)^{\mathbf{T}}\tilde{\mathbf{P}}^{(\mathbf{0})}(\omega)$.

Lemma 2.3 *For every $l \geq 1$,*

$$\begin{aligned} &\tilde{\mathbf{P}}^{(\mathbf{k})}(\omega) - \sum_{q\in L_l^+(\omega)} \tilde{\lambda}_q(\omega)^k \mathcal{P}(q,\omega)\tilde{\mathbf{v}}_{\mathbf{q}}(\omega) \\ &= \tilde{\lambda}_l(\omega)^k \left(\sum_{q\in L_l(\omega)} \mathcal{P}(q,\omega)\tilde{\mathbf{v}}_{\mathbf{q}}(\omega) + \sum_{q\in L_l^-(\omega)} \left(\frac{\tilde{\lambda}_q(\omega)}{\tilde{\lambda}_l(\omega)}\right)^k \mathcal{P}(q,\omega)\tilde{\mathbf{v}}_{\mathbf{q}}(\omega) \right). \end{aligned}$$

Proof Let $\tilde{X}^{(k)}$ be the l_Xth component of $\tilde{\mathbf{P}}^{(\mathbf{k})}$,

$$\begin{aligned} \tilde{\mathbf{P}}^{(\mathbf{k})}(\omega) &= \tilde{\mathrm{M}}(\omega)\tilde{\mathbf{P}}^{(\mathbf{k-1})}(\omega) \\ &= \tilde{\mathrm{M}}(\omega)^k\tilde{\mathbf{P}}^{(\mathbf{0})}(\omega) \\ &= \tilde{\mathrm{V}}(\omega)^{-1}\tilde{\Lambda}(\omega)^k\tilde{\mathrm{V}}(\omega)\tilde{\mathbf{P}}^{(\mathbf{0})}(\omega) \\ &= \left(\sum_{X\in A,B,C,D,\ldots} \tilde{\lambda}_{l_X}(\omega)^k \tilde{\mathbf{v}}_{\mathbf{l_X}}(\omega)\tilde{\mathbf{u}}_{\mathbf{l_X}}^{\mathbf{T}}(\omega) \right) \tilde{\mathbf{P}}^{(\mathbf{0})}(\omega) \\ &= \sum_{X\in A,B,C,D,\ldots} \tilde{\lambda}_{l_X}(\omega)^k \tilde{\mathbf{v}}_{\mathbf{l_X}}(\omega) \left(\tilde{\mathbf{u}}_{\mathbf{l_X}}^{T}(\omega)\tilde{\mathbf{P}}^{(\mathbf{0})}(\omega)\right) \\ &= \sum_{X\in A,B,C,D,\ldots} \tilde{\lambda}_{l_X}(\omega)^k \mathcal{P}(l_X,\omega)\tilde{\mathbf{v}}_{\mathbf{l_X}}(\omega) \end{aligned}$$

which implies the result. ∎

Remark This lemma tells us that the combination

$$\tilde{\lambda}_l(\omega)^k \sum_{q\in L_l(\omega)} \mathcal{P}(q,\omega)\tilde{\mathbf{v}}_{\mathbf{q}}(\omega)$$

is a good estimate of the frequency

$$\tilde{\mathbf{P}}^{(\mathbf{k})}(\omega) - \sum_{q \in L_l^+(\omega)} \tilde{\lambda}_q(\omega)^k \mathcal{P}(q,\omega) \tilde{\mathbf{v}}_{\mathbf{q}}(\omega)$$

as k grows to infinity, in the same way that

$$\sum_{a+b=l} \frac{x^a y^b}{l!} \frac{\partial^l \mathcal{F}}{\partial x^a \partial y^b}(0,0)$$

is a good estimate of the function

$$\mathcal{F}(x,y) - \sum_{a+b<l} \frac{x^a y^b}{(a+b)!} \frac{\partial^{(a+b)} \mathcal{F}}{\partial x^a \partial y^b}(0,0)$$

as (x,y) converges to $(0,0)$.

2.3 Invariances

In this section, we present the invariances that a scheme may have and which simplify the writing of the matrix's, $\tilde{\mathrm{M}}(\omega)$, components as a combination of the components of the subdivision matrix M.

Let $X_h^{(k)}$ be the $l_{(X,h)}$th component of $P^{(k)}$.

Definition The scheme is *rotationally invariant* if

$$M_{l_{(X,j)},1} = M_{l_{(X,q)},1} =: m_{X,1} \ ,$$

$$M_{1,l_{(X,j)}} = M_{1,l_{(X,q)}} =: m_{1,X} \ ,$$

if the mark point is a vertex, and

$$M_{l_{(X,j)},l_{(Y,h)}} = M_{l_{(X,j+q)},l_{(Y,h+q)}} =: m_{(X,Y),j-h}$$

with $m_{(X,Y),h} = m_{(X,Y),h+n}$, whatever the mark point is.

Let $\tilde{X}^{(k)}$ be the l_Xth component of $\tilde{P}^{(k)}$, and $\tilde{M}_{l_X,l_Y}(\omega)$ the components of $\tilde{\mathrm{M}}(\omega)$.

Lemma 2.4 *Consider a rotationally invariant scheme. If the mark point is a vertex, for all* $Y \in \{B, C, D, \ldots\}$,

$$\begin{aligned} \tilde{M}_{1,1}(0) &= M_{1,1} \\ \tilde{M}_{1,l_Y}(0) &= n m_{(1,Y)} \\ \tilde{M}_{l_Y,1}(0) &= m_{Y,1} \ . \end{aligned}$$

Furthermore, whatever the mark point and the rotational frequency ω *are,* $\forall X \in \{B, C, D, \ldots\}$, $\forall Y \in \{B, C, D, \ldots\}$,

$$\tilde{M}_{l_X,l_Y}(\omega) = \sum_{q=1}^{n} m_{(X,Y),q} \exp\left(-\frac{2i\pi\omega}{n}(q + \alpha_X - \alpha_Y + d\alpha) \right) \ .$$

Proof We prove the most general case, where the mark point is a vertex. Proving the dual case follows trivially.

$$\begin{aligned} A^{(k+1)} &= M_{1,1}A^{(k)} + \sum_{Y\in B,C,D,\ldots}\sum_{h=1}^{n} M_{1,l_{(Y,h)}}Y_h^{(k)} \\ &= M_{1,1}A^{(k)} + \sum_{Y\in B,C,D,\ldots}\sum_{h=1}^{n} m_{(1,Y)}Y_h^{(k)} \\ &= M_{1,1}A^{(k)} + \sum_{Y\in B,C,D,\ldots} m_{(1,Y)}\tilde{Y}^{(k)}(0) \end{aligned}$$

So,

$$\begin{aligned} \tilde{A}^{(k+1)} &= n\left(M_{1,1}A^{(k)} + \sum_{Y\in B,C,D,\ldots} m_{(1,Y)}\tilde{Y}^{(k)}(0)\right) \\ &= M_{1,1}\tilde{A}^{(k)} + \sum_{Y\in B,C,D,\ldots} nm_{(1,Y)}\tilde{Y}^{(k)}(0) \end{aligned}$$

Furthermore, for all $X \in \{B, C, D, \ldots\}$, for all $j \in \{1 \ldots n\}$,

$$\begin{aligned} X_j^{(k+1)} &= M_{l_{(X,j)},1}A^{(k)} + \sum_{Y\in B,C,D,\ldots}\sum_{h=1}^{n} M_{l_{(X,j)},l_{(Y,h)}}Y_h^{(k)} \\ &= m_{X,1}A^{(k)} + \sum_{Y\in B,C,D,\ldots}\sum_{h=1}^{n} m_{(X,Y),j-h}Y_h^{(k)} \\ &= m_{X,1}A^{(k)} + \sum_{Y\in B,C,D,\ldots}\sum_{q=j-1}^{j-n} m_{(X,Y),q}Y_{j-q}^{(k)} \end{aligned}$$

But, writing $Y_h^{(k)} = Y_{h+n}^{(k)}$, we get

$$X_j^{(k+1)} = m_{X,1}A^{(k)} + \sum_{Y\in B,C,D,\ldots}\sum_{q=1}^{n} m_{(X,Y),q}Y_{j-q}^{(k)}$$

$$\begin{aligned} \tilde{X}^{(k+1)}(\omega) &= \sum_{j=1}^{n}\left(m_{X,1}A^{(k)} + \sum_{Y\in B,C,D,\ldots}\sum_{q=1}^{n} m_{(X,Y),q}Y_{j-q}^{(k)}\right)\exp\left(-\frac{2i\pi\omega}{n}(j+\alpha_X+\alpha_{k+1})\right) \\ &= m_{X,1}\sum_{j=1}^{n} A^{(k)}\exp\left(-\frac{2i\pi\omega}{n}(j+\alpha_X+\alpha_{k+1})\right) \\ &+ \sum_{Y\in B,C,D,\ldots}\exp\left(-\frac{2i\pi\omega}{n}(\alpha_X-\alpha_Y+\alpha_{k+1}-\alpha_k)\right) \\ &\sum_{q=1}^{n} m_{(X,Y),q}\sum_{j=1}^{n} Y_{j-q}^{(k)}\exp\left(-\frac{2i\pi\omega}{n}(j+\alpha_Y+\alpha_k)\right) \end{aligned}$$

$$
\begin{aligned}
&= m_{X,1}\tilde{A}^{(k)}(0)\delta_{\omega,0} \\
&\quad + \sum_{Y\in B,C,D,\ldots} \exp\left(-\frac{2i\pi\omega}{n}(\alpha_X-\alpha_Y+d\alpha)\right) \\
&\quad \sum_{q=1}^{n} m_{(X,Y),q}\exp\left(-\frac{2i\pi\omega}{n}q\right)\sum_{j=1}^{n} Y_{j-q}^{(k)}\exp\left(-\frac{2i\pi\omega}{n}(j-q+\alpha_Y)\right) \\
&= m_{X,1}\tilde{A}^{(k)}(0)\delta_{\omega,0} \\
&\quad + \sum_{Y\in B,C,D,\ldots} \exp\left(-\frac{2i\pi\omega}{n}(\alpha_X-\alpha_Y+d\alpha)\right)\sum_{q=1}^{n} m_{(X,Y),q}\exp\left(-\frac{2i\pi\omega}{n}q\right)\tilde{Y}^{(k)}
\end{aligned}
$$

which leads to the result. ∎

Remark Whatever the mark point is, the components of the matrix $\tilde{\mathrm{M}}(0)$ are real. Furthermore, if the mark point is a vertex, for all $Y \in \{B, C, D, \ldots\}$, the components $\tilde{M}_{1,1}(0)$ and $\tilde{M}_{1,l_Y}(0)$ are real.

A consequence of this lemma is the following relationship between certain eigenelements of a rotational invariant scheme.

Lemma 2.5 *Consider a scheme with $\tilde{M}(\omega)$, $\omega \in \{0,\ldots,n-1\}$, as transformed subdivision matrices. Let $\tilde{\lambda}_q(\omega)$, $q \in \{0,\ldots,n-1\}$, be the eigenvalues of $\tilde{M}(\omega)$, and $\tilde{v}_q(\omega)$ the related eigenvectors. If the scheme is rotationally invariant then, for all $q \in \{0,\ldots,n-1\}$, $\tilde{\lambda}_q(0)$ and $\tilde{v}_q(0)$ are real. Furthermore, for every $\omega \in \{1,\ldots,n-1\}$, the eigenvalues and eigenvectors of $\tilde{M}(\omega)$ may be sorted such that*

$$
\left\{\begin{array}{lcl}
\tilde{\lambda}_q(\omega) &=& \tilde{\lambda}_q^*(n-\omega)\ ,\ \textit{and} \\
\tilde{v}_q(\omega) &=& \tilde{v}_q^*(n-\omega)\ ,
\end{array}\right.
$$

with c^* being the conjugate of c.

Proof From lemma 2.4, if the scheme is rotationally invariant, then $\tilde{M}(\omega) = \tilde{M}^*(n-\omega)$. Furthermore,

$$
\begin{aligned}
\tilde{M}(n-\omega)\tilde{v}_q(n-\omega) &= \tilde{\lambda}_q(n-\omega)\tilde{v}_q(n-\omega)\ , \\
\tilde{M}^*(n-\omega)\tilde{v}_q^*(n-\omega) &= \tilde{\lambda}_q^*(n-\omega)\tilde{v}_q^*(n-\omega)\ ,
\end{aligned}
$$

as a consequence,

$$
\tilde{M}(\omega)\tilde{v}_q^*(n-\omega) = \tilde{\lambda}_q^*(n-\omega)\tilde{v}_q^*(n-\omega)
$$

which leads to the result. ∎

To introduce mirror invariance, we need some notation. Consider the half of the first sector between the positive x-axis and the sector's centre line. The vertices in this region may be split into three families. The vertices which lie on the x-axis belong to the *basement* family. In Fig. 1(a), B and D are in the basement. In Fig. 1(b), no vertex belongs to the basement. The vertices which lie on the centre line of the sector belong to the *ceiling* family. In Fig. 1(a), C and F belong to the ceiling. In Fig. 1(b), B, D and G belong to the ceiling. The other vertices belong to the *floor* family. In Fig. 1(a), E belongs to the floor. In Fig. 1(b), C, E and F belong to the floor. If the mark point is a vertex, A could belong to the basement or to the ceiling, but we prefer to consider its case separately.

Definition The scheme is *p-mirror invariant*, $p \in \mathbb{Z}$, if

$$m_{1,X} = m_{1,X'} \quad \text{and} \quad m_{X,1} = m_{X',1}$$

and

$$M_{l_{(X,1)},l_{(Y,h)}} = M_{l_{\mathrm{mir}(X,1)},l_{\mathrm{mir}(Y,h-p)}}$$

with

$$\mathrm{mir}(X,h) = \begin{cases} X_{n-h+2} & \text{if } X \text{ belongs to the basement,} \\ X_{n-h+1} & \text{if } X \text{ belongs to the ceiling,} \\ X'_{n-h+1} & \text{if } X \text{ belongs to the floor.} \end{cases}$$

Remark For instance, Loop [8] or Catmull-Clark [4] scheme are rotationally and 0-mirror invariant whereas $\sqrt{3}$ scheme [7] is rotationally and 1-mirror invariant.

Lemma 2.6 *If the scheme is both rotationally and p-mirror invariant, then, depending on the nature of X and Y, $m_{(X,Y),q}$ is equal to*

$X\backslash Y$	*basement*	*ceiling*	*floor*
basement	$m_{(X,Y),-q-p}$	$m_{(X,Y),1-q-p}$	$m_{(X,Y'),1-q-p}$
ceiling	$m_{(X,Y),-1-q-p}$	$m_{(X,Y),-q-p}$	$m_{(X,Y'),-q-p}$
floor	$m_{(X',Y),-1-q-p}$	$m_{(X',Y),-q-p}$	$m_{(X',Y'),-q-p}$

Proof Let X and Y belong to the basement. If the scheme is p-mirror invariant, then

$$M_{l_{(X,1)},l_{(Y,h)}} = M_{l_{(X,1)},l_{(Y,n-h+2+p)}}$$

Furthermore, if the scheme has rotational invariances, then

$$\begin{aligned} m_{(X,Y),1-h} &= m_{(X,Y),1-(n-h+2+p)} \\ &= m_{(X,Y),h-1-p} \end{aligned}$$

Thus,

$$m_{(X,Y),q} = m_{(X,Y),-q-p}$$

Similar arguments are run for the other configurations. ∎

2.4 C^p-Convergence

We propose the following definition for the C^p-convergence of a scheme. The scheme C^p-converges in the vicinity of a mark point if

- for every X in the infinite vicinity $\{B, C, D, \ldots\}$ of A, there exist phases α_X and, for all j in $\{1, \ldots, n\}$, for every k, radii $\varrho_X^{(k)}$ and a C^p-continuous function $\mathcal{F}^{(k)}(x,y)$ such that, if the mark point is a vertex then

$$A^{(k)} = \mathcal{F}^{(k)}(0,0) \ ,$$

and, whatever the mark point is,

$$X_j^{(k)} = \mathcal{F}^{(k)}(\varrho_X^{(k)} \cos(\theta_{(X,j,k)}), \varrho_X^{(k)} \sin(\theta_{(X,j,k)})) \ ,$$

- Furthermore, the sequence of pth differentials $\left(d^p \mathcal{F}^{(k)}\right)_k$ converges uniformly onto $d^p \mathcal{F}$ which is the pth differential of a C^p-continuous parameterisation $\mathcal{F}(x, y)$ of the limit surface in the vicinity of the limit mark point.
- Finally, for all $q \in 0 \ldots p-1$, the sequence $\left(d^q \mathcal{F}^{(k)}(0,0)\right)_k$ converges onto $d^q \mathcal{F}(0,0)$.

In this definition, an infinite vicinity $\{B, C, D, \ldots\}$ is taken into account. In any practical application, we will consider only a finite number of vertices. An intuitive choice is the minimal set of vertices whose linear combination defines the mark point at each subdivision step [17]. This practical restriction is not inconsistent with finding only necessary conditions for C^p-convergence.

From the definition, we see that if the scheme C^p-converges in the vicinity of a mark point, then the sequence of meshes converges towards a C^p-continuous surface around the limit mark point. But the converse is not true: a scheme, which converges towards a C^p-continuous surface is not necessarily C^p-convergent. Note also that the definition domain of $\mathcal{F}^{(k)}$ shrinks as k grows since $\lim_{k\to\infty}(\varrho_X^{(k)}) = 0$ from Sect. 2.1.

2.5 Behaviour of the Limit Points

In Sect. 3 we will be considering the necessary conditions for C^2-convergence. Therefore, consider a scheme which C^2-converges in the vicinity of a mark point. The parameterisation $\mathcal{F}(x, y)$ is C^2-continuous. From its Taylor expansion around $(0, 0)$, we may describe the behaviour of the limit points in the vicinity of the limit mark point. In the following lines, we detail this behaviour with derivatives of the limit function and according to the regularity of the scheme convergence.

Lemma 2.7 *If the scheme C^0-converges then $\forall X \in \{B, C, D, \ldots\}$, $\forall j \in \{1, \ldots, n\}$,*

$$\begin{aligned} \lim_{k\to\infty}(A^{(k)}) &= \mathcal{F}(0,0)\ , && \text{if the mark point is a vertex, and} \\ \lim_{k\to\infty}(X_j^{(k)}) &= \mathcal{F}(0,0)\ , && \text{whatever the mark point is.} \end{aligned}$$

Proof From the definition of C^0-convergence, we know that

$$A^{(k)} = \mathcal{F}^{(k)}(0,0) \text{ and } \lim_{k\to\infty}\left(\mathcal{F}^{(k)}(0,0)\right) = \mathcal{F}(0,0).$$

So,

$$\lim_{k\to\infty}(A^{(k)}) = \mathcal{F}(0,0).$$

Furthermore,

$$\begin{aligned} \|X_j^{(k)} - \mathcal{F}(0,0)\| &= \|\mathcal{F}^{(k)}(\rho_X^{(k)}\cos(\theta_{(X,j,k)}), \|\rho_X^{(k)}\sin(\theta_{(X,j,k)})) - \mathcal{F}(0,0)\| \\ &\leq \|\mathcal{F}^{(k)}(\rho_X^{(k)}\cos(\theta_{(X,j,k)}), \rho_X^{(k)}\sin(\theta_{(X,j,k)})) \\ &- \mathcal{F}(\rho_X^{(k)}\cos(\theta_{(X,j,k)}), \rho_X^{(k)}\sin(\theta_{(X,j,k)}))\| \\ &+ \|\mathcal{F}(\rho_X^{(k)}\cos(\theta_{(X,j,k)}), \rho_X^{(k)}\sin(\theta_{(X,j,k)})) - \mathcal{F}(0,0)\| \end{aligned}$$

From the definition of C^0-convergence, we know that $\mathcal{F}^{(k)}(x,y)$ converges uniformly towards $\mathcal{F}(x,y)$ in the vicinity $\mathcal{V}$ of $(0,0)$:

$$\forall\varepsilon\ \exists K_\varepsilon : k > K_\varepsilon \Rightarrow \forall(x,y)\in\mathcal{V},\ \|\mathcal{F}^{(k)}(x,y)-\mathcal{F}(x,y)\| < \varepsilon$$

Furthermore,

$$\lim_{k\to\infty}(\rho_X^{(k)}) = 0 \qquad i.e. \qquad \forall\alpha\ \exists N_\alpha : n > N_\alpha \Rightarrow \mid \rho_X^{(n)} \mid < \alpha$$

and there exists $\alpha_\mathcal{V}$ such that for all θ

$$\mid \rho \mid < \alpha_\mathcal{V} \Rightarrow (\rho\cos(\theta), \rho\sin(\theta)) \in \mathcal{V}$$

so,

$$k > \max(K_\varepsilon, N_{\alpha_\mathcal{V}}) \Rightarrow$$
$$\|\mathcal{F}^{(k)}(\rho_X^{(k)}\cos(\theta_{(X,j,k)}), \rho_X^{(k)}\sin(\theta_{(X,j,k)})) - \mathcal{F}(\rho_X^{(k)}\cos(\theta_{(X,j,k)}), \rho_X^{(k)}\sin(\theta_{(X,j,k)}))\| < \varepsilon$$

Finally, from the definition of C^0-convergence, we know that $\mathcal{F}(x,y)$ is C^0-continuous in $(0,0)$:

$$\forall\varepsilon\ \exists\alpha_\varepsilon :\mid \rho \mid < \alpha_\varepsilon \Rightarrow \forall\theta \|\mathcal{F}(\rho\cos(\theta), \rho\sin(\theta) - \mathcal{F}(0,0)\| < \varepsilon$$

So,

$$k > \max(K_\varepsilon, N_{\alpha_\mathcal{V}}, N_{\alpha_\varepsilon}) \Rightarrow \|X_j^{(k)} - \mathcal{F}(0,0)\| < 2\varepsilon$$

∎

Lemma 2.8 *If the scheme C^1-converges then $\forall X\in\{B,C,D,\ldots\}$, $\forall j\in\{1,\ldots,n\}$,*

$$\lim_{k\to\infty}\left(\left(\frac{X_j^{(k)}-\mathcal{F}^{(k)}(0,0)}{\varrho_X^{(k)}}\right) - \left(\cos(\theta_{(X,j,k)})\frac{\partial\mathcal{F}}{\partial x}(0,0) + \sin(\theta_{(X,j,k)})\frac{\partial\mathcal{F}}{\partial y}(0,0)\right)\right) = 0\ .$$

Proof We define the following notation:

$$\mathcal{F}_x = \tfrac{\partial\mathcal{F}}{\partial x}(0,0) \quad \mathcal{F}_y = \tfrac{\partial\mathcal{F}}{\partial y}(0,0)$$

$$\begin{aligned}
&\left\|\frac{X_j^{(k)}-\mathcal{F}^{(k)}(0,0)}{\rho_X^{(k)}} - \left(\cos(\theta_{(X,j,k)})\frac{\partial\mathcal{F}}{\partial x}(0,0) + \sin(\theta_{(X,j,k)})\frac{\partial\mathcal{F}}{\partial y}(0,0)\right)\right\| \\
= &\left\|\frac{\mathcal{F}^{(k)}(\rho_X^{(k)}\cos(\theta_{(X,j,k)}), \rho_X^{(k)}\sin(\theta_{(X,j,k)})) - \mathcal{F}^{(k)}(0,0)}{\rho_X^{(k)}} - \left(\cos(\theta_{(X,j,k)})\mathcal{F}_x + \sin(\theta_{(X,j,k)})\mathcal{F}_y\right)\right\| \\
\leq &\left\|\frac{\mathcal{F}^{(k)}(\rho_X^{(k)}\cos(\theta_{(X,j,k)}), \rho_X^{(k)}\sin(\theta_{(X,j,k)})) - \mathcal{F}(\rho_X^{(k)}\cos(\theta_{(X,j,k)}), \rho_X^{(k)}\sin(\theta_{(X,j,k)}))}{\rho_X^{(k)}}\right. \\
&\left. - \frac{\mathcal{F}^{(k)}(0,0)-\mathcal{F}(0,0)}{\rho_X^{(k)}}\right\| \\
&+ \left\|\frac{\mathcal{F}(\rho_X^{(k)}\cos(\theta_{(X,j,k)}), \rho_X^{(k)}\sin(\theta_{(X,j,k)})) - \mathcal{F}(0,0)}{\rho_X^{(k)}} - \left(\cos(\theta_{(X,j,k)})\mathcal{F}_x + \sin(\theta_{(X,j,k)})\mathcal{F}_y\right)\right\|
\end{aligned}$$

From the definition of C^1-convergence, we know that $d\mathcal{F}^{(k)}(x,y)$ converges uniformly towards $d\mathcal{F}(x,y)$ in the vicinity $\mathcal{V}$ of $(0,0)$:

$$\forall\varepsilon\ \exists K_\varepsilon : k > K_\varepsilon \Rightarrow \forall(x,y)\in\mathcal{V},\ \forall\theta,$$

$$\left\|\left(\cos(\theta)\frac{\partial\mathcal{F}^{(k)}}{\partial x}(x,y)+\sin(\theta)\frac{\partial\mathcal{F}^{(k)}}{\partial y}(x,y)\right)-\left(\cos(\theta)\frac{\partial\mathcal{F}}{\partial x}(x,y)+\sin(\theta)\frac{\partial\mathcal{F}}{\partial y}(x,y)\right)\right\|<\varepsilon$$

Applying the Taylor-Lagrange inequality on $(x,y)\mapsto\mathcal{F}^{(k)}(x,y)-\mathcal{F}(x,y)$, we get for all $(\rho\cos(\theta),\rho\sin(\theta))$ in $\mathcal{V}$,

$$\left\|\left(\mathcal{F}^{(k)}(\rho\cos(\theta),\rho\sin(\theta))-\mathcal{F}(\rho\cos(\theta),\rho\sin(\theta))\right)-\left(\mathcal{F}^{(k)}(0,0)-\mathcal{F}(0,0)\right)\right\|<\varepsilon\mid\rho\mid$$

In particular,

$$k>\max(K_\varepsilon,N_{\alpha_\mathcal{V}})\Rightarrow$$

$$\left\|\frac{\mathcal{F}^{(k)}(\rho_X^{(k)}\cos(\theta_{(X,j,k)}),\rho_X^{(k)}\sin(\theta_{(X,j,k)}))-\mathcal{F}(\rho_X^{(k)}\cos(\theta_{(X,j,k)}),\rho_X^{(k)}\sin(\theta_{(X,j,k)}))}{\rho_X^{(k)}}\right.$$
$$\left.-\frac{\mathcal{F}^{(k)}(0,0)-\mathcal{F}(0,0)}{\rho_X^{(k)}}\right\|<\varepsilon$$

Finally, from the definition of C^1-convergence, we know that $\mathcal{F}(x,y)$ is C^1-continuous in $(0,0)$:

$$\forall\varepsilon\ \exists\alpha_\varepsilon :\mid\rho\mid<\alpha_\varepsilon\Rightarrow\forall\theta$$

$$\left\|\frac{\mathcal{F}(\rho\cos(\theta),\rho\sin(\theta))-\mathcal{F}}{\rho}-(\cos(\theta)\mathcal{F}_x+\sin(\theta)\mathcal{F}_y)\right\|<\varepsilon$$

So,

$$k>\max(K_\varepsilon,N_{\alpha_\mathcal{V}},N_{\alpha_\varepsilon})\Rightarrow$$

$$\left\|\frac{X_j^{(k)}-\mathcal{F}^{(k)}(0,0)}{\rho_X^{(k)}}-\left(\cos(\theta_{(X,j,k)})\frac{\partial\mathcal{F}}{\partial x}(0,0)+\sin(\theta_{(X,j,k)})\frac{\partial\mathcal{F}}{\partial y}(0,0)\right)\right\|<2\varepsilon$$

∎

Lemma 2.9 *If the scheme C^2-converges then $\forall X\in\{B,C,D,\ldots\}$, $\forall j\in\{1,\ldots,n\}$,*

$$\lim_{k\to\infty}\left(\frac{\Delta_{X,j}^{(k)}}{{\varrho_X^{(k)}}^2}-\left[\left(\frac{\partial^2\mathcal{F}}{\partial x^2}(0,0)+\frac{\partial^2\mathcal{F}}{\partial y^2}(0,0)\right)\frac{1}{4}+\frac{\partial^2\mathcal{F}}{\partial x\partial y}(0,0)\frac{\sin(2\theta_{(X,j,k)})}{2}\right.\right.$$
$$\left.\left.+\left(\frac{\partial^2\mathcal{F}}{\partial x^2}(0,0)-\frac{\partial^2\mathcal{F}}{\partial y^2}(0,0)\right)\frac{\cos(2\theta_{(X,j,k)})}{4}\right]\right)=0\ .$$

with

$$\Delta_{X,j}^{(k)}:=X_j^{(k)}-\mathcal{F}^{(k)}(0,0)-\varrho_X^{(k)}\left(\cos(\theta_{(X,j,k)})\frac{\partial\mathcal{F}^{(k)}}{\partial x}(0,0)-\sin(\theta_{(X,j,k)})\frac{\partial\mathcal{F}^{(k)}}{\partial y}(0,0)\right)\ .$$

Proof We define the following notation:

$$\mathcal{F}_{xx} = \tfrac{\partial^2 \mathcal{F}}{\partial x^2}(0,0) \quad \mathcal{F}_{yy} = \tfrac{\partial^2 \mathcal{F}}{\partial y^2}(0,0) \quad \mathcal{F}_{xy} = \tfrac{\partial^2 \mathcal{F}}{\partial x \partial y}(0,0)$$

$$\begin{aligned}
&\Bigg\| \frac{X_j^{(k)} - \mathcal{F}^{(k)}(0,0) - \rho_X^{(k)} \cos(\theta_{(X,j,k)}) \frac{\partial \mathcal{F}^{(k)}}{\partial x}(0,0) - \rho_X^{(k)} \sin(\theta_{(X,j,k)}) \frac{\partial \mathcal{F}^{(k)}}{\partial y}(0,0)}{{\rho_X^{(k)}}^2} \\
&- \left[\left(\frac{\partial^2 \mathcal{F}}{\partial x^2}(0,0) + \frac{\partial^2 \mathcal{F}}{\partial y^2}(0,0) \right) \frac{1}{4} + \right. \\
&\left. \left(\frac{\partial^2 \mathcal{F}}{\partial x^2}(0,0) - \frac{\partial^2 \mathcal{F}}{\partial y^2}(0,0) \right) \frac{\cos(2\theta_{(X,j,k)})}{4} + \frac{\partial^2 \mathcal{F}}{\partial x \partial y}(0,0) \frac{\sin(2\theta_{(X,j,k)})}{2} \right] \Bigg\| \\
= \; &\Bigg\| \frac{\mathcal{F}^{(k)}(\rho_X^{(k)} \cos(\theta_{(X,j,k)}), \rho_X^{(k)} \sin(\theta_{(X,j,k)})) - \mathcal{F}^{(k)}(0,0)}{{\rho_X^{(k)}}^2} \\
&\frac{-\rho_X^{(k)} \cos(\theta_{(X,j,k)}) \mathcal{F}_x^{(k)} - \rho_X^{(k)} \sin(\theta_{(X,j,k)}) \mathcal{F}_y^{(k)}}{{\rho_X^{(k)}}^2} \\
&- \left[(\mathcal{F}_{xx} + \mathcal{F}_{yy}) \frac{1}{4} + (\mathcal{F}_{xx} - \mathcal{F}_{yy}) \frac{\cos(2\theta_{(X,j,k)})}{4} + \mathcal{F}_{xy} \frac{\sin(2\theta_{(X,j,k)})}{2} \right] \Bigg\| \\
\leq \; &\Bigg\| \frac{\left[\mathcal{F}^{(k)}(\rho_X^{(k)} \cos(\theta_{(X,j,k)}), \rho_X^{(k)} \sin(\theta_{(X,j,k)})) - \mathcal{F}(\rho_X^{(k)} \cos(\theta_{(X,j,k)}), \rho_X^{(k)} \sin(\theta_{(X,j,k)})) \right]}{{\rho_X^{(k)}}^2} \\
&+ \frac{- \left[\mathcal{F}^{(k)}(0,0) - \mathcal{F}(0,0) \right] - \rho_X^{(k)} \cos(\theta_{(X,j,k)}) \left[\mathcal{F}_x^{(k)} - \mathcal{F}_x \right] - \rho_X^{(k)} \sin(\theta_{(X,j,k)}) \left[\mathcal{F}_y^{(k)} - \mathcal{F}_y \right]}{{\rho_X^{(k)}}^2} \Bigg\| \\
&+ \Bigg\| \frac{\mathcal{F}(\rho_X^{(k)} \cos(\theta_{(X,j,k)}), \rho_X^{(k)} \sin(\theta_{(X,j,k)})) - \mathcal{F}(0,0) - \rho_X^{(k)} \cos(\theta_{(X,j,k)}) \mathcal{F}_x - \rho_X^{(k)} \sin(\theta_{(X,j,k)}) \mathcal{F}_y}{{\rho_X^{(k)}}^2} \\
&- \left[(\mathcal{F}_{xx} + \mathcal{F}_{yy}) \frac{1}{4} + (\mathcal{F}_{xx} - \mathcal{F}_{yy}) \frac{\cos(2\theta_{(X,j,k)})}{4} + \mathcal{F}_{xy} \frac{\sin(2\theta_{(X,j,k)})}{2} \right] \Bigg\|
\end{aligned}$$

From the definition of C^2-convergence, we know that $d^2\mathcal{F}^{(k)}(x,y)$ converges uniformly towards $d^2\mathcal{F}(x,y)$ in the vicinity $\mathcal{V}$ of $(0,0)$:

$$\forall \varepsilon \ \exists K_\varepsilon : k > K_\varepsilon \Rightarrow \forall (x,y) \in \mathcal{V}, \ \forall \theta,$$

$$\begin{aligned}
&\Bigg\| \left[\left(\frac{\partial^2 \mathcal{F}^{(k)}}{\partial x^2}(x,y) + \frac{\partial^2 \mathcal{F}^{(k)}}{\partial y^2}(x,y) \right) \frac{1}{4} + \right. \\
&\left. \left(\frac{\partial^2 \mathcal{F}^{(k)}}{\partial x^2}(x,y) - \frac{\partial^2 \mathcal{F}^{(k)}}{\partial y^2}(x,y) \right) \frac{\cos(2\theta)}{4} + \frac{\partial^2 \mathcal{F}^{(k)}}{\partial x \partial y}(x,y) \frac{\sin(2\theta)}{2} \right] \\
&- \left[\left(\frac{\partial^2 \mathcal{F}}{\partial x^2}(x,y) + \frac{\partial^2 \mathcal{F}}{\partial y^2}(x,y) \right) \frac{1}{4} + \right. \\
&\left. \left(\frac{\partial^2 \mathcal{F}}{\partial x^2}(x,y) - \frac{\partial^2 \mathcal{F}}{\partial y^2}(x,y) \right) \frac{\cos(2\theta)}{4} + \frac{\partial^2 \mathcal{F}}{\partial x \partial y}(x,y) \frac{\sin(2\theta)}{2} \right] \Bigg\| < \varepsilon
\end{aligned}$$

Applying the Taylor-Lagrange inequality on $(x, y) \mapsto \mathcal{F}^{(k)}(x, y) - \mathcal{F}(x, y)$, we get for all $(\rho\cos(\theta), \rho\sin(\theta))$ in $\mathcal{V}$,

$$\left\| \left[\mathcal{F}^{(k)}(\rho\cos(\theta), \rho\sin(\theta)) - \mathcal{F}(\rho\cos(\theta), \rho\sin(\theta))\right] - \left[\mathcal{F}^{(k)}(0,0) - \mathcal{F}(0,0)\right] - \rho\cos(\theta)\left[\mathcal{F}_x^{(k)} - \mathcal{F}_x\right] - \rho\sin(\theta)\left[\mathcal{F}_y^{(k)} - \mathcal{F}_y\right] \right\| < \varepsilon\frac{|\,\rho\,|^2}{2}$$

In particular,

$$k > \max(K_\varepsilon, N_{\alpha_\mathcal{V}}) \Rightarrow$$

$$\left\| \frac{\left[\mathcal{F}^{(k)}(\rho_X^{(k)}\cos(\theta_{(X,j,k)}), \rho_X^{(k)}\sin(\theta_{(X,j,k)})) - \mathcal{F}(\rho_X^{(k)}\cos(\theta_{(X,j,k)}), \rho_X^{(k)}\sin(\theta_{(X,j,k)}))\right]}{{\rho_X^{(k)}}^2} + \frac{-\left[\mathcal{F}^{(k)}(0,0) - \mathcal{F}(0,0)\right] - \rho_X^{(k)}\cos(\theta_{(X,j,k)})\left[\mathcal{F}_x^{(k)} - \mathcal{F}_x\right] - \rho_X^{(k)}\sin(\theta_{(X,j,k)})\left[\mathcal{F}_y^{(k)} - \mathcal{F}_y\right]}{{\rho_X^{(k)}}^2} \right\| < \frac{\varepsilon}{2}$$

Finally, from the definition of C^2-convergence, we know that $\mathcal{F}(x, y)$ is C^2-continuous in $(0, 0)$:

$$\forall\varepsilon\ \exists\alpha_\varepsilon :|\,\rho\,|< \alpha_\varepsilon \Rightarrow$$

$$\left\| \frac{\mathcal{F}(\rho\cos(\theta), \rho\sin(\theta)) - \mathcal{F}(0,0) - \rho\cos(\theta)\mathcal{F}_x - \rho\sin(\theta)\mathcal{F}_y}{\rho^2} - \left[(\mathcal{F}_{xx} + \mathcal{F}_{yy})\frac{1}{4} + (\mathcal{F}_{xx} - \mathcal{F}_{yy})\frac{\cos(2\theta)}{4} + \mathcal{F}_{xy}\frac{\sin(2\theta)}{2}\right] \right\| < \varepsilon$$

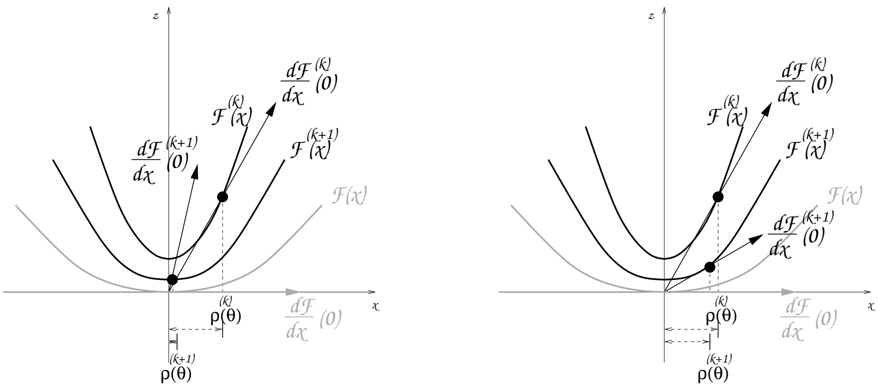
So,

$$k > \max(K_\varepsilon, N_{\alpha_\mathcal{V}}, N_{\alpha_\varepsilon}) \Rightarrow \forall\theta$$

$$\left\| \frac{X_j^{(k)} - \mathcal{F}^{(k)}(0,0) - \rho_X^{(k)}\cos(\theta_{(X,j,k)})\frac{\partial\mathcal{F}^{(k)}}{\partial x}(0,0) - \rho_X^{(k)}\sin(\theta_{(X,j,k)})\frac{\partial\mathcal{F}^{(k)}}{\partial y}(0,0)}{{\rho_X^{(k)}}^2} - \left[\left(\frac{\partial^2\mathcal{F}}{\partial x^2}(0,0) + \frac{\partial^2\mathcal{F}}{\partial y^2}(0,0)\right)\frac{1}{4} + \left(\frac{\partial^2\mathcal{F}}{\partial x^2}(0,0) - \frac{\partial^2\mathcal{F}}{\partial y^2}(0,0)\right)\frac{\cos(2\theta_{(X,j,k)})}{4} + \frac{\partial^2\mathcal{F}}{\partial x\partial y}(0,0)\frac{\sin(2\theta_{(X,j,k)})}{2}\right] \right\| < \frac{3\varepsilon}{2}$$

∎

Remark Note that the three lemmas tell us that the difference between two terms shrinks onto zero as k grows to infinity. But this does not mean in general that both terms converge towards the same limit. For example, in the case of the Kobbelt's $\sqrt{3}$ scheme [7], $\theta_{(X,j,k)} = \frac{2\pi}{n}(j + \alpha_X + k/2)$ does not converge as k grows to infinity.



(a) Bad sampling for approximating the first derivative

(b) Good sampling for approximating the first derivative

Figure 2: Sampling for derivative approximation

2.6 Good Sampling is Necessary

Lemmas (2.7), (2.8) and (2.9) tell us that the derivatives of the limit surface $\mathcal{F}$ may be written as the limit of linear combinations of samplings of the interpolating functions $\mathcal{F}^{(k)}$ with their derivatives on $(0,0)$. That means that we have to know the values of $\mathcal{F}^{(k)}$ and its derivatives on $(0,0)$. In practice, we can replace them by the value of the limit function $\mathcal{F}$ and its derivatives on $(0,0)$. But, to do so, the samplings have to fulfil the following conditions:

$$\lim_{k\to\infty}\left(\frac{\left|\frac{\partial^{a+b}\mathcal{F}^{(k)}}{\partial x^a\partial y^b}(0,0)-\frac{\partial^{a+b}\mathcal{F}}{\partial x^a\partial y^b}(0,0)\right|}{{\varrho^{(k)}}^{a+b+1}}\right)=0 \tag{2}$$

which means that the radial parameters $\varrho^{(k)}$ of the samples must not to shrink more quickly than the functions $\mathcal{F}^{(k)}$ converge to the limit surface.

For instance, we must have

$$\lim_{k\to\infty}\left(\frac{\left|\mathcal{F}^{(k)}(0,0)-\mathcal{F}(0,0)\right|}{\varrho^{(k)}}\right)=0$$

if we want to take

$$\frac{\mathcal{F}^{(k)}(\varrho^{(k)}\cos(\theta),\varrho^{(k)}\sin(\theta))-\mathcal{F}(0,0)}{\varrho^{(k)}}$$

as an approximation of

$$\cos(\theta)\frac{\partial\mathcal{F}}{\partial x}(0,0)+\sin(\theta)\frac{\partial\mathcal{F}}{\partial y}(0,0)$$

as illustrated in Fig. 2.

Another consequence of a bad sampling is a bad sorting of the eigenvalues among the subdivision matrices in the frequency domain. Because the components of $\mathbf{P^{(k)}}$ are samples of the function $\mathcal{F}^{(k)}$, the successive components $\sum_{q\in L_l(\omega)}\mathcal{P}(q,\omega)\tilde{\mathbf{v}}_{\mathbf{q}}(\omega)$ may be interpreted as the frequency of the successive sets

$$\{\mathcal{F}(0,0)\}\ ,$$

$$\left\{\frac{\mathcal{F}^{(k)}(\varrho_X^{(k)}\cos(\theta_{(X,j,k)}),\varrho_X^{(k)}\sin(\theta_{(X,j,k)}))-\mathcal{F}(0,0)}{\varrho_X^{(k)}}\right\}\ ,$$

$$\left\{\frac{\mathcal{F}^{(k)}(\varrho_X^{(k)}\cos(\theta_{(X,j,k)}),\varrho_X^{(k)}\sin(\theta_{(X,j,k)}))}{{\varrho_X^{(k)}}^2}\right.$$
$$\left.-\frac{\mathcal{F}(0,0)+\varrho_X^{(k)}\cos(\theta_{(X,j,k)})\frac{\partial\mathcal{F}}{\partial x}(0,0)+\varrho_X^{(k)}\sin(\theta_{(X,j,k)})\frac{\partial\mathcal{F}}{\partial y}(0,0)}{{\varrho_X^{(k)}}^2}\right\}$$

and so on. So, the non-null frequency components of these successive sets are the same as those of the successive dominant eigenvalues among all the subdivision matrices in the frequency domain. The non-null frequency components of the successive derivatives of the limit surface $\mathcal{F}$ at $(0,0)$ are successively the frequency of a position (0), a tangent plane (± 1), a quadric—a cup (0) or a saddle (± 2)—and so on. If the sampling is good, the frequencies of these successive sets follow the frequencies of the successive derivatives of the limit surface. If the sampling is bad, these sets are bad approximations to the derivatives of the limit surface $\mathcal{F}$ at $(0,0)$ and the successive main eigenvalues do not come from the expected frequencies.

For instance, if the sampling is such that

$$\lim_{k\to\infty}\left(\frac{\left|\mathcal{F}^{(k)}(0,0)-\mathcal{F}(0,0)\right|}{\varrho^{(k)}}\right)\neq 0$$

as illustrated in Fig. 2(a), the samples are too close to each other in comparison with the distance between them and the limit point $\mathcal{F}(0,0)$. Then the frequency of the set of vectors $\mathcal{F}^{(k)}(x,y)-\mathcal{F}(0,0)$, which is the frequency associated with the sub-dominant eigenvalue among all the subdivision matrices in the frequency domain, is the frequency of a point rather than the frequency of a plane. As a consequence, the sub-dominant eigenvalues do not come from the frequencies ± 1 as expected but from the frequency 0 (see lemma 3.3).

In summary, if the sampling is bad, on the one hand, we cannot say anything about the convergence of the scheme, and on the other hand, the successive main eigenvalues do not come from the expected frequency matrices. A simple way to overcome this problem is to ask the successive interpolation $\mathcal{F}^{(k)}$ to interpolate $\mathcal{F}$ with its derivatives at $(0,0)$, which adds an extra constraint.

3 Necessary Conditions for C^2-Convergence and Derivatives of the Limit Surface

Lemmas (2.7), (2.8) and (2.9) describe the behaviour of the limit points. Applying the Discrete Fourier Transform on these equations gives a description of the limit frequencies.

Consistency between this description and the one given by lemma (2.3) implies necessary conditions for the C^2-convergence of the scheme. It gives also the partial derivatives of the limit surface at the mark point. As notation, we say that if $\tilde{X}^{(k)}(\omega)$ is the mth component of $\tilde{\mathbf{P}}^{(\mathbf{k})}(\omega)$, then $(\tilde{v}_l(\omega))_X$ is the mth component of $\tilde{\mathbf{v}}_\mathbf{l}(\omega)$. We assume without any restriction that for every fixed ω, $\tilde{\lambda}_2(\omega)$ is the eigenvalue of $\tilde{\mathrm{M}}(\omega)$ with the greatest modulus after $\tilde{\lambda}_1(\omega)$ and possibly other eigenvalues with same modulus as $\tilde{\lambda}_1(\omega)$: for all ω, $L_1(\omega) = L_2^+(\omega)$. Finally, we choose the phases ϕ_k, introduced in the shifted discrete Fourier transform in Sect. 2.2, to be written as $\phi_k = kd\phi$ with $d\phi$ independent from k.

3.1 C^0-Convergence

Lemma 3.1 *If the scheme C^0-converges, then*

$$\begin{cases} \tilde{\lambda}_1(0) = 1 \ , \\ \left|\tilde{\lambda}_1(\omega)\right| < 1 \quad \textit{for } \omega \neq 0. \end{cases}$$

and if $L_1(0) = \{1\}$, then

$$(\tilde{v}_1(0))_X = \nu_0$$

with ν_0 being a constant, and

$$\mathcal{F}(0,0) = \frac{\mathcal{P}(1,0)}{n}\,(\tilde{v}_1(0))_X \ .$$

Proof From lemma 2.7, we know that if the scheme C^0-converges in the vicinity $\{B, C, D, \ldots\}$ of a mark point, then there exists a function $\mathcal{F}(x, y)$ such that

$$\lim_{k\to\infty} (A^{(k)}) = \mathcal{F}(0,0), \text{ if the mark point is a vertex}$$

and $\forall X \in \{B, C, D, \ldots\}$, $\forall j \in \{1, \ldots, n\}$,

$$\lim_{k\to\infty} (X_j^{(k)}) = \mathcal{F}(0,0) = \mathcal{F} \text{ whatever the mark point is.}$$

From lemma 2.2, we get $\forall X \in \{A, B, C, D, \ldots\}$, $\forall \omega \in \{0 \ldots n-1\}$,

$$\lim_{k\to\infty} (\tilde{X}^{(k)}(\omega)) = \mathrm{DFT}(\mathcal{F}(0,0))(\omega) \ .$$

From lemma 2.1, taking $\Omega = 0$ and $a = \mathcal{F}$, we get

$$\mathrm{DFT}(\mathcal{F}(0,0))(\omega) = n\mathcal{F}(0,0)\delta_{\omega,0} \ .$$

From lemma 2.3, having supposed that for all q and ω, $\left|\tilde{\lambda}_q(\omega)\right| \leq \left|\tilde{\lambda}_1(\omega)\right|$ we get

$$\lim_{k\to\infty} (\tilde{X}^{(k)}(\omega)) = \lim_{k\to\infty} (\tilde{\lambda}_1(\omega)^k \sum_{q\in L_1(\omega)} \mathcal{P}(q,\omega)\,(\tilde{v}_q(\omega))_X) \ .$$

As a consequence,

$$\lim_{k\to\infty} (\tilde{\lambda}_1(\omega)^k \sum_{q\in L_1(\omega)} \mathcal{P}(q,\omega)\,(\tilde{v}_q(\omega))_X) = n\mathcal{F}(0,0)\delta_{\omega,0}.$$

Then,

$$\begin{cases} \tilde{\lambda}_1(0) = 1 & , \\ \left|\tilde{\lambda}_1(\omega)\right| < 1 & \text{for } \omega \neq 0. \end{cases}$$

And

$$\mathcal{F}(0,0) = \frac{1}{n} \sum_{q\in L_1(0)} \mathcal{P}(q,0)\,(\tilde{v}_q(0))_X\,.$$

In particular, if $L_1(0) = \{1\}$, $\forall X \in \{A, B, C, D, \ldots\}$,

$$(\tilde{v}_1(0))_X = \nu_0$$

with ν_0 be a constant, and then

$$\mathcal{F}(0,0) = \frac{\mathcal{P}(1,0)}{n}\,(\tilde{v}_1(0))_X\,.$$

∎

Remark Not only do we get necessary conditions on eigenvalues and eigenvectors of $\tilde{\mathrm{M}}(\omega)$, but we also get the value of $\mathcal{F}(0,0)$, that is the limit mark point.

3.2 C^1-Convergence

Lemma 3.2 *If the scheme C^1-converges, then when k is large, if $L_1(1) = L_1(-1) = \{1\}$, the moduli of the eigencomponents $|(\tilde{v}_1(1))_X|$ and $|(\tilde{v}_1(-1))_X|$ are sorted like the radii of the rings ϱ_X. Furthermore, if $\frac{\partial\mathcal{F}}{\partial x}(0,0) + \mathrm{i}\frac{\partial\mathcal{F}}{\partial y}(0,0) \neq 0$, the phases $\alpha_k = kd\alpha$ and α_X must satisfy*

$$d\alpha = d\phi \pm \frac{n}{2\pi}\varphi_{(\tilde{\lambda}_1(\pm 1))_X} \text{ and} \tag{3}$$

$$\alpha_X = \phi_X \pm \left(\varphi_{(\tilde{v}_1(\pm 1))_X} + \varphi_{\mathcal{P}(1,\pm 1)} \pm \frac{\partial\mathcal{F}}{\partial y}(0,0)/\frac{\partial\mathcal{F}}{\partial x}(0,0)\right)\,. \tag{4}$$

Proof From lemma 2.8, we know that if the scheme C^1-converges in the vicinity $\{B, C, D, \ldots\}$ of a mark point, then there exist functions $\mathcal{F}^{(k)}(x,y)$ and $\mathcal{F}(x,y)$ such that $\forall X \in \{B, C, D, \ldots\}$, $\forall j \in \{1, \ldots, n\}$,

$$\lim_{k\to\infty}\left(\frac{X_j^{(k)} - \mathcal{F}^{(k)}(0,0)}{\rho_X^{(k)}} - \cos(\theta_{(X,j,k)})\frac{\partial\mathcal{F}}{\partial x}(0,0) - \sin(\theta_{(X,j,k)})\frac{\partial\mathcal{F}}{\partial y}(0,0)\right) = 0\,.$$

From lemma 2.2, we get $\forall X \in \{B, C, D, \ldots\}$, $\forall\omega \in \{0, \ldots, n-1\}$,

$$\begin{aligned} &\lim_{k\to\infty}\left(\mathrm{DFT}\left(\frac{X_j^{(k)} - \mathcal{F}^{(k)}(0,0)}{\rho_X^{(k)}}\right)(\omega)\right. \\ &- \left.\mathrm{DFT}\left(\cos(\theta_{(X,j,k)})\frac{\partial\mathcal{F}}{\partial x}(0,0) + \sin(\theta_{(X,j,k)})\frac{\partial\mathcal{F}}{\partial y}(0,0)\right)(\omega)\right) = 0\,. \end{aligned}$$

From equation (1) $\theta_{(X,j,k)} = \frac{2\pi}{n}(j + \alpha_X + \alpha_k)$ so from lemma 2.1, taking $\alpha = \alpha_X + \alpha_k$ and on the one hand $\Omega = 0$ and $a = \mathcal{F}^{(k)}$, and on the other hand $\Omega = 1$ and $a = \mathcal{F}_x$, $b = \mathcal{F}_y$, we get

$$\lim_{k\to\infty}\left(\frac{\tilde{X}_j^{(k)}(\omega) - n\mathcal{F}^{(k)}\delta_{\omega,0}}{\rho_X^{(k)}} - n\frac{\mathcal{F}_x \mp \mathrm{i}\mathcal{F}_y}{2}\exp\left(\pm\frac{2\mathrm{i}\pi}{n}(\alpha_X + \alpha_k - \phi_X - \phi_k)\right)\right) = 0\ .$$

From lemma 2.3, having supposed that $L_1(\omega) = L_2^+(\omega)$ we get with $\omega = \pm 1$,

$$\frac{\tilde{X}^{(k)}(\omega)}{\rho_X^{(k)}} = \frac{\tilde{\lambda}_1(\omega)^k}{\rho_X^{(k)}}\left(\sum_{q\in L_1(\omega)}\mathcal{P}(q,\omega)\left(\tilde{v}_q(\omega)\right)_X + \sum_{q\in L_1^-(\omega)}\left(\frac{\tilde{\lambda}_q(\omega)}{\tilde{\lambda}_1(\omega)}\right)^k\mathcal{P}(q,\omega)\left(\tilde{v}_q(\omega)\right)_X\right) \quad (5)$$

So,

$$\lim_{k\to\infty}\left(\frac{\tilde{\lambda}_1(\pm 1)^k}{\rho_X^{(k)}}\sum_{q\in L_1(\pm 1)}\mathcal{P}(q,\pm 1)\left(\tilde{v}_q(\pm 1)\right)_X - n\frac{\mathcal{F}_x \mp \mathrm{i}\mathcal{F}_y}{2}\exp\left(\pm\frac{2\mathrm{i}\pi}{n}(\alpha_X + \alpha_k - \phi_X - \phi_k)\right)\right) = 0\ . \quad (6)$$

As a consequence

$$\lim_{k\to\infty}\left|\frac{\tilde{\lambda}_1(\pm 1)^k}{\rho_X^{(k)}}\sum_{q\in L_1(\pm 1)}\mathcal{P}(q,\pm 1)\left(\tilde{v}_q(\pm 1)\right)_X\right| = \frac{n}{2}\left|\mathcal{F}_x \mp \mathrm{i}\mathcal{F}_y\right|$$

which leads to,

$$\rho_X^{(k)} = \mathcal{A}_{X,1}(k)\left|\tilde{\lambda}_1(1)\right|^k\left|\sum_{q\in L_1(1)}\mathcal{P}(q,1)\left(\tilde{v}_q(1)\right)_X\right|\frac{2}{n}/\left|\mathcal{F}_x - \mathrm{i}\mathcal{F}_y\right| \quad (7)$$

$$= \mathcal{A}_{X,-1}(k)\left|\tilde{\lambda}_1(-1)\right|^k\left|\sum_{q\in L_1(-1)}\mathcal{P}(q,-1)\left(\tilde{v}_q(-1)\right)_X\right|\frac{2}{n}/\left|\mathcal{F}_x + \mathrm{i}\mathcal{F}_y\right| \quad (8)$$

where

$$\lim_{k\to\infty}\mathcal{A}_{X,1}(k) = \lim_{k\to\infty}\mathcal{A}_{X,-1}(k) = 1$$

In particular, if $L_1(1) = L_1(-1) = \{1\}$,

$$\rho_X^{(k)} = \left|\tilde{\lambda}_1(1)\right|^k\frac{1}{\nu_{X,1}(k)}\left|\left(\tilde{v}_1(1)\right)_X\right|$$

$$= \left|\tilde{\lambda}_1(-1)\right|^k\frac{1}{\nu_{X,-1}(k)}\left|\left(\tilde{v}_1(-1)\right)_X\right|$$

where

$$\lim_{k\to\infty} \nu_{X,1}(k) = \nu_1 = \frac{n}{2}\frac{|\mathcal{F}_x - \mathrm{i}\mathcal{F}_y|}{|\mathcal{P}(1,1)|}$$

$$\lim_{k\to\infty} \nu_{X,-1}(k) = \nu_{-1} = \frac{n}{2}\frac{|\mathcal{F}_x + \mathrm{i}\mathcal{F}_y|}{|\mathcal{P}(1,-1)|}$$

This implies that when k is large, the moduli of the eigencomponents $|(\tilde{v}_1(1))_X|$ and $|(\tilde{v}_1(-1))_X|$ are sorted as the parameters ρ_X.

From equation (6) we get also, if $\frac{\partial\mathcal{F}}{\partial x}(0,0)+\mathrm{i}\frac{\partial\mathcal{F}}{\partial y}(0,0) \neq 0$, and if $L_1(1) = L_1(-1) = \{1\}$,

$$\lim_{k\to\infty}\left(k\varphi_{\tilde{\lambda}_1(\pm1)} + \varphi_{\mathcal{P}(1,\pm1)} + \varphi_{(\tilde{v}_1(\pm1))_X} - \left(\mp\frac{\mathcal{F}_y}{\mathcal{F}_x} \pm \frac{2\pi}{n}\left(\alpha_X + \alpha_k - \phi_X - \phi_k\right)\right)\right) = 0$$

$$\begin{aligned}
\lim_{k\to\infty}\left(k\varphi_{\tilde{\lambda}_1(\pm1)} \mp \frac{2\pi}{n}\left(\alpha_k - \phi_k\right)\right) &= -\varphi_{\mathcal{P}(1,\pm1)} - \varphi_{(\tilde{v}_1(\pm1))_X} \mp \frac{\mathcal{F}_y}{\mathcal{F}_x} \pm \frac{2\pi}{n}\left(\alpha_X - \phi_X\right)\\
\lim_{k\to\infty}\left(k\left(\varphi_{\tilde{\lambda}_1(\pm1)} \mp \frac{2\pi}{n}\left(d\alpha - d\phi\right)\right)\right) &= -\varphi_{\mathcal{P}(1,\pm1)} - \varphi_{(\tilde{v}_1(\pm1))_X} \mp \frac{\mathcal{F}_y}{\mathcal{F}_x} \pm \frac{2\pi}{n}\left(\alpha_X - \phi_X\right)
\end{aligned}$$

which leads to

$$\begin{cases} \varphi_{\tilde{\lambda}_1(\pm1)} &= \pm\frac{2\pi}{n}\left(d\alpha - d\phi\right)\\ \varphi_{(\tilde{v}_1(\pm1))_X} &= \mp\frac{\mathcal{F}_y}{\mathcal{F}_x} - \varphi_{\mathcal{P}(1,\pm1)} \pm \frac{2\pi}{n}\left(\alpha_X - \phi_X\right)\end{cases}$$

which is equivalent to

$$\begin{cases} d\alpha &= d\phi \pm \frac{n}{2\pi}\varphi_{\left(\tilde{\lambda}_1(\pm1)\right)_X} \text{ and}\\ \alpha_X &= \phi_X \pm \frac{n}{2\pi}\left(\varphi_{(\tilde{v}_1(\pm1))_X} + \varphi_{\mathcal{P}(1,\pm1)} \pm \frac{\partial\mathcal{F}}{\partial y}(0,0)/\frac{\partial\mathcal{F}}{\partial x}(0,0)\right)\,.\end{cases}$$

∎

Remark Equations (3) and (4) imply the following relationship between $d\alpha$, α_X and $d\phi$, ϕ_X:

$$\begin{cases} d\alpha = d\phi & \Leftrightarrow \quad \tilde{\lambda}_1(\pm1) \text{ are real}\\ \alpha_X = \phi_X & \Leftrightarrow \quad \varphi_{(\tilde{v}_1(\pm1))_X} \text{ do not depend on } X.\end{cases}$$

This means that if we get real eigenvalues, and components of eigenvectors with the same phase, then we have chosen phases ϕ_X and ϕ_k for the shifted discrete Fourier transform, equal to the intrinsic phases α_X and α_k of the scheme.

If the scheme is rotationally invariant, we get from this lemma a possible practical definition for the radii ϱ_X and also the values of the partial differences $\frac{\partial\mathcal{F}}{\partial x}(0,0)$ and $\frac{\partial\mathcal{F}}{\partial y}(0,0)$. Indeed,we know from lemma 2.5 that $\left|\tilde{\lambda}_1(1)\right| = \left|\tilde{\lambda}_1(-1)\right|$. So for every k,

$$\frac{|(\tilde{v}_1(1))_X|}{|(\tilde{v}_1(-1))_X|} = \frac{\nu_{X,1}(k)}{\nu_{X,-1}(k)}\,.$$

As a consequence,

$$\frac{|(\tilde{v}_1(1))_X|}{|(\tilde{v}_1(-1))_X|} = \frac{|\mathcal{P}(1,-1)|}{|\mathcal{P}(1,1)|} \; .$$

And because this equation is true for every X and we can scale the eigenvectors (with consequential effects on the left eigenvectors), we can get

$$\frac{|(\tilde{v}_1(1))_X|}{|(\tilde{v}_1(-1))_X|} = \frac{|\mathcal{P}(1,-1)|}{|\mathcal{P}(1,1)|} = 1 \; ,$$

and so

$$\nu_1 = \nu_{-1} \; .$$

For simplicity, we can define the radii ϱ_X as follows (implying $\nu_1 = \nu_{-1} = 1$),

$$\varrho_X^{(k)} = \left|\tilde{\lambda}_1(1)\right|^k |(\tilde{v}_1(1))_X| = \left|\tilde{\lambda}_1(-1)\right|^k |(\tilde{v}_1(-1))_X| \; .$$

However the radii ϱ_X are defined, if the scheme is rotationally invariant, then

$$\begin{aligned}
\left|\frac{\partial \mathcal{F}}{\partial x}(0,0) \mp \mathrm{i}\frac{\partial \mathcal{F}}{\partial y}(0,0)\right| &= \frac{2}{n}\,|\mathcal{P}(1,\pm 1)| \\
\frac{\partial \mathcal{F}}{\partial x}(0,0) \mp \mathrm{i}\frac{\partial \mathcal{F}}{\partial y}(0,0) &= \frac{2}{n}\,|\mathcal{P}(1,\pm 1)| \exp\left(\mathrm{i}\left(\varphi_{(\tilde{v}_1(\pm 1))_X} + \varphi_{\mathcal{P}(1,\pm 1)} \mp \frac{2\pi}{n}\left(\alpha_X - \phi_X\right)\right)\right) \\
&= \frac{2}{n}\mathcal{P}(1,\pm 1) \exp\left(\mathrm{i}\left(\varphi_{(\tilde{v}_1(\pm 1))_X} \mp \frac{2\pi}{n}\left(\alpha_X - \phi_X\right)\right)\right)
\end{aligned}$$

Furthermore, if we define α_X as $\alpha_X = \phi_X \pm \frac{n}{2\pi}\varphi_{(\tilde{v}_1(\pm 1))_X}$, (or if $\varphi_{(\tilde{v}_1(\pm 1))_X}$ does not depend on X, as $\alpha_X = \phi_X$ after having scaled the eigenvectors $(\tilde{v}_1(\pm 1))_X$ to be real) then

$$\frac{\partial \mathcal{F}}{\partial x}(0,0) \mp \mathrm{i}\frac{\partial \mathcal{F}}{\partial y}(0,0) = \frac{2}{n}\mathcal{P}(1,\pm 1) \; .$$

which is equivalent to

$$\left\{\begin{aligned}
\tfrac{\partial \mathcal{F}}{\partial x}(0,0) &= \tfrac{2}{n}\,\Re\left(\mathcal{P}(1,1)\right) = \tfrac{2}{n}\,\Re\left(\mathcal{P}(1,-1)\right) \; , \\
\tfrac{\partial \mathcal{F}}{\partial y}(0,0) &= \tfrac{2}{n}\,\Im\left(\mathcal{P}(1,1)\right) = -\tfrac{2}{n}\,\Im\left(\mathcal{P}(1,-1)\right) \; .
\end{aligned}\right.$$

Lemma 3.3 *If the scheme C^1-converges and the mark point is a vertex, then*

$$\left|\tilde{\lambda}_2(0)\right| < \left|\tilde{\lambda}_1(\pm 1)\right| \; .$$

If the scheme C^1-converges and the mark point is a face centre, then

$$\left|\tilde{\lambda}_2(0)\right| < \left|\tilde{\lambda}_1(\pm 1)\right| \qquad \textit{iff} \qquad \lim_{k\to\infty}\left(\frac{\mathcal{F}^{(k)}(0,0) - \mathcal{F}(0,0)}{\left|\tilde{\lambda}_1(\pm 1)\right|^k}\right) = 0 \; .$$

Proof From equation (6) and lemma 2.3, having supposed that $L_1(\omega) = L_2^+(\omega)$ we get with $\omega = 0$,

$$\frac{\tilde{X}^{(k)}(0) - n\mathcal{F}(0,0)}{\rho_X^{(k)}} = \frac{\tilde{X}^{(k)}(0) - \tilde{\lambda}_1(0)^k \sum_{q\in L_2^+(0)} \mathcal{P}(q,0)\left(\tilde{v}_q(0)\right)_X}{\rho_X^{(k)}} =$$

$$\frac{\tilde{\lambda}_2(0)^k}{\rho_X^{(k)}}\left(\sum_{q\in L_2(0)} \mathcal{P}(q,0)\left(\tilde{v}_q(0)\right)_X + \sum_{q\in L_2^-(0)}\left(\frac{\tilde{\lambda}_q(0)}{\tilde{\lambda}_2(0)}\right)^k \mathcal{P}(q,0)\left(\tilde{v}_q(0)\right)_X\right)$$

If the mark point is a vertex, then

$$\tilde{A}^{(k)}(0) = nA^{(k)} = n\mathcal{F}^{(k)}(0,0)$$

So,

$$\begin{aligned}
0 &= \lim_{k\to\infty}\left(\frac{\tilde{X}^{(k)}(0) - n\mathcal{F}^{(k)}(0,0)}{\rho_X^{(k)}}\right)\\
0 &= \lim_{k\to\infty}\left(\frac{(\tilde{X}^{(k)}(0) - n\mathcal{F}(0,0)) - (\tilde{A}^{(k)}(0) - n\mathcal{F}(0,0))}{\rho_X^{(k)}}\right)\\
0 &= \lim_{k\to\infty}\left(\frac{\tilde{\lambda}_2(0)^k}{\rho_X^{(k)}}\Big(\sum_{q\in L_2(0)} \mathcal{P}(q,0)\left[\left(\tilde{v}_q(0)\right)_X - \left(\tilde{v}_q(0)\right)_A\right]\right.\\
&\quad \left. + \sum_{q\in L_2^-(0)}\left(\frac{\tilde{\lambda}_q(0)}{\tilde{\lambda}_2(0)}\right)^k \mathcal{P}(q,0)\left[\left(\tilde{v}_q(0)\right)_X - \left(\tilde{v}_q(0)\right)_A\right]\Big)\right)\\
0 &= \lim_{k\to\infty}\left(\frac{\tilde{\lambda}_2(0)^k}{\rho_X^{(k)}}\sum_{q\in L_2(0)} \mathcal{P}(q,0)\left[\left(\tilde{v}_q(0)\right)_X - \left(\tilde{v}_q(0)\right)_A\right]\right)
\end{aligned}$$

Then, with lemma 3.2,

$$\left|\tilde{\lambda}_2(0)\right| < \left|\tilde{\lambda}_1(\pm 1)\right| \;.$$

If the mark point is a face centre, then

$$\begin{aligned}
0 &= \lim_{k\to\infty}\left(\frac{\tilde{X}^{(k)}(0) - n\mathcal{F}^{(k)}(0,0)}{\rho_X^{(k)}}\right)\\
0 &= \lim_{k\to\infty}\left(\frac{(\tilde{X}^{(k)}(0) - n\mathcal{F}(0,0)) - (n\mathcal{F}^{(k)}(0,0) - n\mathcal{F}(0,0))}{\rho_X^{(k)}}\right)\\
0 &= \lim_{k\to\infty}\left(\frac{\tilde{\lambda}_2(0)^k}{\rho_X^{(k)}}\Big(\sum_{q\in L_2(0)} \mathcal{P}(q,0)\left(\tilde{v}_q(0)\right)_X + \sum_{q\in L_2^-(0)}\left(\frac{\tilde{\lambda}_q(0)}{\tilde{\lambda}_2(0)}\right)^k \mathcal{P}(q,0)\left(\tilde{v}_q(0)\right)_X\Big)\right.\\
&\quad \left. -n\frac{\mathcal{F}^{(k)}(0,0) - \mathcal{F}(0,0))}{\rho_X^{(k)}}\right)\\
0 &= \lim_{k\to\infty}\left(\frac{\tilde{\lambda}_2(0)^k}{\rho_X^{(k)}}\sum_{q\in L_2(0)} \mathcal{P}(q,0)\left(\tilde{v}_q(0)\right)_X - n\frac{\mathcal{F}^{(k)}(0,0) - \mathcal{F}(0,0))}{\rho_X^{(k)}}\right)
\end{aligned}$$

Then, with lemma 3.2,

$$\left|\tilde{\lambda}_2(0)\right| < \left|\tilde{\lambda}_1(\pm 1)\right| \qquad \text{iff} \qquad \lim_{k\to\infty}\left(\frac{\mathcal{F}^{(k)}(0,0)-\mathcal{F}(0,0)}{\left|\tilde{\lambda}_1(\pm 1)\right|^k}\right) = 0\ .$$

∎

Remark If the mark point is a face centre, we do not control $\mathcal{F}^{(k)}(0,0)$. So, as explained in Sect. 2.6, the sampling may be inadequate for our analysis.

Lemma 3.4 *If the scheme C^1-converges, and if $\omega \notin \{-1,0,1\}$, then*

$$\left|\tilde{\lambda}_1(\omega)\right| < \left|\tilde{\lambda}_1(\pm 1)\right|\ .$$

Proof From equation (5) and lemma 2.3, having supposed that $L_1(\omega) = L_2^+(\omega)$ we get with $\omega \notin \{-1,0,1\}$,

$$\frac{\tilde{X}^{(k)}(\omega)}{\rho_X^{(k)}} = \frac{\tilde{\lambda}_1(\omega)^k}{\rho_X^{(k)}}\left(\sum_{q\in L_1(\omega)}\mathcal{P}(1,\omega)\left(\tilde{v}_1(\omega)\right)_X + \sum_{q\in L_1^-(\omega)}\left(\frac{\tilde{\lambda}_q(\omega)}{\tilde{\lambda}_1(\omega)}\right)^k\mathcal{P}(q,\omega)\left(\tilde{v}_q(\omega)\right)_X\right)$$

So,

$$\begin{aligned} 0 &= \lim_{k\to\infty}\left(\frac{\tilde{\lambda}_1(\omega)^k}{\rho_X^{(k)}}\left(\sum_{q\in L_1(\omega)}\mathcal{P}(1,\omega)\left(\tilde{v}_1(\omega)\right)_X + \sum_{q\in L_1^-(\omega)}\left(\frac{\tilde{\lambda}_q(\omega)}{\tilde{\lambda}_1(\omega)}\right)^k\mathcal{P}(q,\omega)\left(\tilde{v}_q(\omega)\right)_X\right)\right) \\ &= \lim_{k\to\infty}\left(\frac{\tilde{\lambda}_1(\omega)^k}{\rho_X^{(k)}}\right) \end{aligned}$$

Then, with equations (7) and (8),

$$\left|\tilde{\lambda}_1(\omega)\right| < \left|\tilde{\lambda}_1(\pm 1)\right|.$$

∎

3.3 C^2-Convergence

Lemma 3.5 *If the scheme C^2-converges and the mark point is a vertex, then*

$$\tilde{\lambda}_2(0) = \tilde{\lambda}_1(\pm 1)^2\ ,$$

and, if $L_2(0) = \{2\}$, then

$$\frac{\left(\tilde{v}_2(0)\right)_X - \left(\tilde{v}_2(0)\right)_A}{\left(\tilde{v}_1(1)\right)_X^2} \qquad \text{and} \qquad \frac{\left(\tilde{v}_2(0)\right)_X - \left(\tilde{v}_2(0)\right)_A}{\left(\tilde{v}_1(-1)\right)_X^2}$$

depend neither on X nor on k.

If the scheme C^2-converges and the mark point is a face centre, then

$$\tilde{\lambda}_2(0) = \left|\tilde{\lambda}_1(\pm 1)\right|^2$$

iff

$$\lim_{k\to\infty}\left(\frac{\mathcal{F}^{(k)}(0,0)-\mathcal{F}(0,0)}{{\rho_X^{(k)}}^2}\right) = \frac{\nu_{\pm 1}^2}{n}\sum_{q\in L_2(0)}\mathcal{P}(q,0)\frac{(\tilde{v}_q(0))_X}{\left|(\tilde{v}_1(\pm 1))_X\right|^2} - \frac{\mathcal{F}_{xx}+\mathcal{F}_{yy}}{4}$$

Proof From lemma 2.9, we know that if the scheme C^2-converges in the vicinity $\{B, C, D, \ldots\}$ of a mark point, then there exist function $\mathcal{F}^{(k)}$ and $\mathcal{F}(x,y)$ such that $\forall X \in \{B, C, D, \ldots\}$, $\forall j \in \{1, \ldots, n\}$,

$$\lim_{k\to\infty}\left(\left[\frac{X_j^{(k)} - \mathcal{F}^{(k)}(0,0) - \rho_X^{(k)}\cos(\theta_{(X,j,k)})\frac{\partial\mathcal{F}^{(k)}}{\partial x}(0,0) - \rho_X^{(k)}\sin(\theta_{(X,j,k)})\frac{\partial\mathcal{F}^{(k)}}{\partial y}(0,0)}{{\rho_X^{(k)}}^2}\right] - \left[\left(\frac{\partial^2\mathcal{F}}{\partial x^2}(0,0)+\frac{\partial^2\mathcal{F}}{\partial y^2}(0,0)\right)\frac{1}{4} + \left(\frac{\partial^2\mathcal{F}}{\partial x^2}(0,0)-\frac{\partial^2\mathcal{F}}{\partial y^2}(0,0)\right)\frac{\cos(2\theta_{(X,j,k)})}{4} + \frac{\partial^2\mathcal{F}}{\partial x\partial y}(0,0)\frac{\sin(2\theta_{(X,j,k)})}{2}\right]\right) = 0\ .$$

From lemma 2.2, we get $\forall X \in \{A, B, C, D, \ldots\}$, $\forall \omega \in \{0 \ldots n-1\}$,

$$\lim_{k\to\infty}\left(\mathrm{DFT}\left(\frac{X_j^{(k)} - \mathcal{F}^{(k)}(0,0) - \rho_X^{(k)}\cos(\theta_{(X,j,k)})\frac{\partial\mathcal{F}^{(k)}}{\partial x}(0,0) - \rho_X^{(k)}\sin(\theta_{(X,j,k)})\frac{\partial\mathcal{F}^{(k)}}{\partial y}(0,0)}{{\rho_X^{(k)}}^2}\right)(\omega) - \mathrm{DFT}\left(\left(\frac{\partial^2\mathcal{F}}{\partial x^2}(0,0)+\frac{\partial^2\mathcal{F}}{\partial y^2}(0,0)\right)\frac{1}{4}\right)(\omega) - \mathrm{DFT}\left(\left(\frac{\partial^2\mathcal{F}}{\partial x^2}(0,0)-\frac{\partial^2\mathcal{F}}{\partial y^2}(0,0)\right)\frac{\cos(2\theta_{(X,j,k)})}{4} + \frac{\partial^2\mathcal{F}}{\partial x\partial y}(0,0)\frac{\sin(2\theta_{(X,j,k)})}{2}\right)(\omega)\right) = 0\ . \tag{9}$$

From lemma 2.1, taking $\alpha = \alpha_X + \alpha_k$, and $\Omega = 0$, and $a = (\mathcal{F}_{xx} + \mathcal{F}_{yy})/4$, we get with $\omega = 0$

$$\lim_{k\to\infty}\left(\frac{\tilde{X}^{(k)}(0) - n\mathcal{F}^{(k)}(0,0)}{{\rho_X^{(k)}}^2}\right) = \frac{n(\mathcal{F}_{xx}+\mathcal{F}_{yy})}{4}$$

From lemma 2.3, having supposed that $L_1(\omega) = L_2^+(\omega)$ we get

$$\frac{\tilde{X}^{(k)}(0) - n\mathcal{F}(0,0)}{\rho_X^{(k)}} = \frac{\tilde{X}^{(k)}(0) - \tilde{\lambda}_1(0)^k \sum_{q\in L_2^+(0)}\mathcal{P}(q,0)\,(\tilde{v}_q(0))_X}{{\rho_X^{(k)}}^2} =$$

$$\frac{\tilde{\lambda}_2(0)^k}{{\rho_X^{(k)}}^2}\left(\sum_{q\in L_2(0)}\mathcal{P}(q,0)\,(\tilde{v}_q(0))_X + \sum_{q\in L_2^-(0)}\left(\frac{\tilde{\lambda}_q(0)}{\tilde{\lambda}_2(0)}\right)^k\mathcal{P}(q,0)\,(\tilde{v}_q(0))_X\right)$$

If the mark point is a vertex, then

$$\tilde{A}^{(k)}(0) = nA^{(k)} = n\mathcal{F}^{(k)}(0,0)$$

So,

$$\begin{aligned}
\frac{n(\mathcal{F}_{xx}+\mathcal{F}_{yy})}{4} &= \lim_{k\to\infty}\left(\frac{\tilde{X}^{(k)}(0)-\tilde{A}^{(k)}(0)}{{\rho_X^{(k)}}^2}\right)\\
&= \lim_{k\to\infty}\left(\frac{(\tilde{X}^{(k)}(0)-n\mathcal{F})-(\tilde{A}^{(k)}(0)-n\mathcal{F})}{{\rho_X^{(k)}}^2}\right)\\
&= \lim_{k\to\infty}\left(\frac{\tilde{\lambda}_2(0)^k}{{\rho_X^{(k)}}^2}(\sum_{q\in L_2(0)}\mathcal{P}(q,0)\left[(\tilde{v}_q(0))_X-(\tilde{v}_q(0))_A\right]\right.\\
&\qquad\left.+\sum_{q\in L_2^-(0)}\left(\frac{\tilde{\lambda}_q(0)}{\tilde{\lambda}_2(0)}\right)^k\mathcal{P}(q,0)\left[(\tilde{v}_q(0))_X-(\tilde{v}_q(0))_A\right])\right)\\
&= \lim_{k\to\infty}\left(\frac{\tilde{\lambda}_2(0)^k}{{\rho_X^{(k)}}^2}\sum_{q\in L_2(0)}\mathcal{P}(q,0)\left[(\tilde{v}_q(0))_X-(\tilde{v}_q(0))_A\right]\right)\\
&= \lim_{k\to\infty}\left(\left(\frac{\tilde{\lambda}_2(0)}{\left|\tilde{\lambda}_1(\pm1)\right|^2}\right)^k\nu_{X,\pm1}(k)^2\sum_{q\in L_2(0)}\mathcal{P}(q,0)\frac{\left[(\tilde{v}_q(0))_X-(\tilde{v}_q(0))_A\right]}{\left|(\tilde{v}_1(\pm1))_X\right|^2}\right)
\end{aligned}$$

and because $\lim_{k\to\infty}\nu_{X,\pm1}(k)=\nu_{\pm1}$,

$$\tilde{\lambda}_2(0)=\left|\tilde{\lambda}_1(\pm1)\right|^2,$$

and

$$\sum_{q\in L_2(0)}\mathcal{P}(q,0)\frac{\left[(\tilde{v}_q(0))_X-(\tilde{v}_q(0))_A\right]}{\left|(\tilde{v}_1(\pm1))_X\right|^2}$$

does not depend on X.

In particular, if $L_2(0)=\{2\}$,

$${\nu_{\pm1}}^2\frac{(\tilde{v}_2(0))_X-(\tilde{v}_2(0))_A}{\left|(\tilde{v}_1(\pm1))_X\right|^2}=\nu_{20}$$

and

$$\mathcal{F}_{xx}+\mathcal{F}_{yy}=4{\nu_{\pm1}}^2\nu_{20}\frac{\mathcal{P}(2,0)}{n}$$

If the mark point is a face centre,

$$\frac{n(\mathcal{F}_{xx}+\mathcal{F}_{yy})}{4} = \lim_{k\to\infty}\left(\frac{(\tilde{X}^{(k)}(0)-n\mathcal{F}(0,0))-(n\mathcal{F}^{(k)}(0,0)-n\mathcal{F}(0,0))}{{\rho_X^{(k)}}^2}\right)$$

$$= \lim_{k\to\infty}\left(\frac{\tilde{\lambda}_2(0)^k}{\rho_X^{(k)^2}}(\sum_{q\in L_2(0)}\mathcal{P}(q,0)\left(\tilde{v}_q(0)\right)_X+\sum_{q\in L_2^-(0)}\left(\frac{\tilde{\lambda}_q(0)}{\tilde{\lambda}_2(0)}\right)^k\mathcal{P}(q,0)\left(\tilde{v}_q(0)\right)_X)\right.$$

$$\left.-n\frac{\mathcal{F}^{(k)}(0,0)-\mathcal{F}(0,0)}{\rho_X^{(k)^2}}\right)$$

$$= \lim_{k\to\infty}\left(\frac{\tilde{\lambda}_2(0)^k}{\rho_X^{(k)^2}}\sum_{q\in L_2(0)}\mathcal{P}(q,0)\left(\tilde{v}_q(0)\right)_X-n\frac{\mathcal{F}^{(k)}(0,0)-\mathcal{F}(0,0)}{\rho_X^{(k)^2}}\right)$$

$$= \lim_{k\to\infty}\left(\left(\frac{\tilde{\lambda}_2(0)}{\left|\tilde{\lambda}_1(\pm1)\right|^2}\right)^k\nu_{X,\pm1}(k)^2\sum_{q\in L_2(0)}\mathcal{P}(q,0)\frac{\left(\tilde{v}_q(0)\right)_X}{\left|\left(\tilde{v}_1(\pm1)\right)_X\right|^2}\right.$$

$$\left.-n\frac{\mathcal{F}^{(k)}(0,0)-\mathcal{F}(0,0)}{\rho_X^{(k)^2}}\right)$$

and because $\lim_{k\to\infty}\nu_{X,\pm1}(k)=\nu_{\pm1}$,

$$\tilde{\lambda}_2(0)=\left|\tilde{\lambda}_1(\pm1)\right|^2$$

iff

$$\lim_{k\to\infty}\left(\frac{\mathcal{F}^{(k)}(0,0)-\mathcal{F}(0,0)}{\rho_X^{(k)^2}}\right)=\frac{\nu_{\pm1}^2}{n}\sum_{q\in L_2(0)}\mathcal{P}(q,0)\frac{\left(\tilde{v}_q(0)\right)_X}{\left|\left(\tilde{v}_1(\pm1)\right)_X\right|^2}-\frac{\mathcal{F}_{xx}+\mathcal{F}_{yy}}{4}\,.$$

∎

Remark If the mark point is a vertex, and if we define $\varrho_X^{(k)}$ as proposed in the remark given after lemma 3.2, then, if $L_2(0)=\{2\}$, we obtain

$$\frac{\left(\tilde{v}_2(0)\right)_X-\left(\tilde{v}_2(0)\right)_A}{\left|\left(\tilde{v}_1(\pm1)\right)_X\right|^2}=\nu_{20}$$

and

$$\frac{\partial^2\mathcal{F}}{\partial x^2}(0,0)+\frac{\partial^2\mathcal{F}}{\partial y^2}(0,0)=4\nu_{20}\frac{\mathcal{P}(2,0)}{n}\,.$$

If the mark point is a face centre, we would find great advantage in asking for all k, $\mathcal{F}^{(k)}(0,0)$ to be equal to $\mathcal{F}(0,0)$ which is known to be

$$\mathcal{F}(0,0)=\frac{\mathcal{P}(1,0)}{n}\left(\tilde{v}_1(0)\right)_X\,.$$

from lemma 3.1. Indeed, in this case, we would get the same results as for a mark point being equal to a vertex (see lemma 3.3 and lemma 3.5).

Furthermore, if the mark point is a face centre, we do not control $\mathcal{F}^{(k)}(0,0)$. So, as explained in Sect. 2.6, the sampling may be inadequate for our analysis.

Lemma 3.6 *If the scheme C^2-converges, then*

$$\left|\tilde{\lambda}_1(\pm 2)\right| = \left|\tilde{\lambda}_1(\pm 1)\right|^2 \ ,$$

and if $L_1(2) = L_1(-2) = \{1\}$, then each of the ratios

$$\frac{|(\tilde{v}_1(2))_X|}{|(\tilde{v}_1(1))_X|^2} \ , \qquad \frac{|(\tilde{v}_1(2))_X|}{|(\tilde{v}_1(-1))_X|^2} \ , \qquad \frac{|(\tilde{v}_1(-2))_X|}{|(\tilde{v}_1(1))_X|^2} \ , \qquad \textit{and} \qquad \frac{|(\tilde{v}_1(-2))_X|}{|(\tilde{v}_1(-1))_X|^2}$$

does not depend on X. Furthermore, if $\frac{\partial^2\mathcal{F}}{\partial x^2}(0,0) - \frac{\partial^2\mathcal{F}}{\partial y^2}(0,0) \mp \mathrm{i}2\frac{\partial^2\mathcal{F}}{\partial x\partial y}(0,0) \neq 0$,

$$\tilde{\lambda}_1(\pm 2) = \tilde{\lambda}_1(\pm 1)^2 \ ,$$

and

$$\varphi_{(\tilde{v}_1(\pm 2))_X} = \mp 2\frac{\frac{\partial^2\mathcal{F}}{\partial x\partial y}(0,0)}{\left(\frac{\partial^2\mathcal{F}}{\partial x^2}(0,0) - \frac{\partial^2\mathcal{F}}{\partial y^2}(0,0)\right)} - \varphi_{\mathcal{P}(1,\pm 2)} + 2\left(\varphi_{(\tilde{v}_1(\pm 1))_X} \pm \frac{\frac{\partial\mathcal{F}}{\partial y}(0,0)}{\frac{\partial\mathcal{F}}{\partial y}(0,0)} + \varphi_{\mathcal{P}(1,\pm 1)}\right) \ .$$

Proof From equation (9) and lemma 2.1, taking $\alpha = \alpha_X + \alpha_k$, $\Omega = 2$ and $a = (\mathcal{F}_{xx} - \mathcal{F}_{yy})/4$, $b = (\mathcal{F}_{xy})/2$,

$$\lim_{k\to\infty}\left(\frac{\tilde{X}^{(k)}(\pm 2)}{\rho_X^{(k)2}} - n\frac{\mathcal{F}_{xx} - \mathcal{F}_{yy} \mp i2\mathcal{F}_{xy}}{8}\exp\left(\pm\frac{4\pi\mathrm{i}}{n}\left(\alpha_X + \alpha_k - \phi_X - \phi_k\right)\right)\right) = 0 \ .$$

From lemma 2.3, having supposed that $L_1(\omega) = L_2^+(\omega)$ we get

$$\frac{\tilde{X}^{(k)}(\omega)}{\rho_X^{(k)2}} =$$

$$\frac{\tilde{\lambda}_1(\omega)^k}{\rho_X^{(k)2}}\left(\sum_{q\in L_1(\omega)}\mathcal{P}(q,\omega)\left(\tilde{v}_q(\omega)\right)_X + \sum_{q\in L_1^-(\omega)}\left(\frac{\tilde{\lambda}_q(\omega)}{\tilde{\lambda}_1(\omega)}\right)^k\mathcal{P}(q,\omega)\left(\tilde{v}_q(\omega)\right)_X\right) \ .$$

So,

$$\begin{aligned}\lim_{k\to\infty}&\left(\frac{\tilde{\lambda}_1(\pm 2)^k}{\rho_X^{(k)2}}\sum_{q\in L_1(\pm 2)}\mathcal{P}(q,\pm 2)\left(\tilde{v}_q(\pm 2)\right)_X\right.\\ &\left.-n\frac{\mathcal{F}_{xx} - \mathcal{F}_{yy} \mp i2\mathcal{F}_{xy}}{8}\exp\left(\pm\frac{4\pi\mathrm{i}}{n}\left(\alpha_X + \alpha_k - \phi_X - \phi_k\right)\right)\right) = 0 \ . \qquad (10)\end{aligned}$$

This implies that

$$\lim_{k\to\infty}\left|\frac{\tilde{\lambda}_1(\pm 2)^k}{\tilde{\lambda}_1(\pm 1)^{2k}}\sum_{q\in L_1(\pm 2)}\mathcal{P}(q,\pm 2)\frac{(\tilde{v}_q(\pm 2))_X}{|(\tilde{v}_q(\pm 1))_X|^2}\nu_{X,\pm 1}^2(k)\right| = \frac{n}{8}\left|\mathcal{F}_{xx} - \mathcal{F}_{yy} \mp i2\mathcal{F}_{xy}\right| \ .$$

And because

$$\lim_{k\to\infty} \nu_{X,\pm1}(k) = \nu_{\pm1}$$

we get

$$\left|\tilde{\lambda}_1(\pm2)\right| = \left|\tilde{\lambda}_1(\pm1)\right|^2 \ ,$$

and

$$\left|\sum_{q\in L_1(\pm2)} \mathcal{P}(q,\pm2)\frac{(\tilde{v}_q(\pm2))_X}{|(\tilde{v}_1(\pm1))_X|^2}\right|$$

does not depend on X.

In particular, if $L_1(\pm2) = \{1\}$,

$$\frac{|(\tilde{v}_1(2))_X|}{|(\tilde{v}_1(1))_X|^2} = \frac{\nu_{21}}{\nu_1} \ , \qquad \frac{|(\tilde{v}_1(2))_X|}{|(\tilde{v}_1(-1))_X|^2} = \frac{\nu_{21}}{\nu_{-1}} \ ,$$

$$\frac{|(\tilde{v}_1(-2))_X|}{|(\tilde{v}_1(1))_X|^2} = \frac{\nu_{-21}}{\nu_1} \ , \qquad \frac{|(\tilde{v}_1(-2))_X|}{|(\tilde{v}_1(-1))_X|^2} = \frac{\nu_{-21}}{\nu_{-1}}$$

with

$$\nu_{21} = \frac{n}{8}\frac{|\mathcal{F}_{xx} - \mathcal{F}_{yy} - i2\mathcal{F}_{xy}|}{|\mathcal{P}(1,2)|}$$

and

$$\nu_{-21} = \frac{n}{8}\frac{|\mathcal{F}_{xx} - \mathcal{F}_{yy} + \mathrm{i}2\mathcal{F}_{xy}|}{|\mathcal{P}(1,-2)|}$$

which leads to the result.

From equation (10), we get also if $\mathcal{F}_{xx} - \mathcal{F}_{yy} \mp \mathrm{i}2\mathcal{F}_{xy} \neq 0$, and if $L_1(\pm2) = \{1\}$,

$$\lim_{k\to\infty}\Big(k\varphi_{\tilde{\lambda}_1(\pm2)} + \varphi_{\mathcal{P}(1,\pm2)} + \varphi_{(\tilde{v}_1(\pm2))_X} - \left(\mp\frac{2\mathcal{F}_{xy}}{\mathcal{F}_{xx} - \mathcal{F}_{yy}} \pm \frac{4\pi}{n}\left(\alpha_X + \alpha_k - \phi_X - \phi_k\right)\right)\Big) = 0$$

and because $\alpha_k = kd\alpha$ and $\phi_k = kd\phi$,

$$\begin{cases} \varphi_{\tilde{\lambda}_1(\pm2)} & = \pm\frac{4\pi}{n}(d\alpha - d\phi) \\ \varphi_{(\tilde{v}_1(\pm2))_X} & = \mp\frac{2\mathcal{F}_{xy}}{\mathcal{F}_{xx}-\mathcal{F}_{yy}} - \varphi_{\mathcal{P}(1,\pm2)} \pm \frac{4\pi}{n}(\alpha_X - \phi_X) \end{cases}$$

So, from lemma 3.2 we get

$$\varphi_{\tilde{\lambda}_1(\pm2)} = 2\varphi_{\tilde{\lambda}_1(\pm1)}$$

which leads to

$$\tilde{\lambda}_1(\pm2) = \tilde{\lambda}_1(\pm1)^2 \ ,$$

and also

$$\varphi_{(\tilde{v}_1(\pm2))_X} = \mp2\frac{\frac{\partial^2\mathcal{F}}{\partial x\partial y}(0,0)}{\left(\frac{\partial^2\mathcal{F}}{\partial x^2}(0,0) - \frac{\partial^2\mathcal{F}}{\partial y^2}(0,0)\right)} - \varphi_{\mathcal{P}(1,\pm2)} + 2\left(\varphi_{(\tilde{v}_1(\pm1))_X} \pm \frac{\frac{\partial\mathcal{F}}{\partial y}(0,0)}{\frac{\partial\mathcal{F}}{\partial y}(0,0)} + \varphi_{\mathcal{P}(1,\pm1)}\right) \ .$$

∎

Remark If the scheme is rotationally invariant, we know from lemma 2.5 that

$$|(\tilde{v}_1(2))_X| = |(\tilde{v}_1(-2))_X|$$

so,

$$\nu_{21} = \nu_1 \frac{|(\tilde{v}_1(2))_X|}{|(\tilde{v}_1(1))_X|^2} = \nu_1 \frac{|(\tilde{v}_1(-2))_X|}{|(\tilde{v}_1(1))_X|^2} = \nu_{-21} \; .$$

And we know from the remark after lemma 3.2, that $\nu_1 = \nu_{-1}$. Furthermore, if, as in the same remark, we define α_X as $\alpha_X = \phi_X \pm \frac{n}{2\pi}\varphi_{(\tilde{v}_1(\pm 1))_X}$, then the difference of phases

$$\varphi_{(\tilde{v}_1(\pm 2))_X} - 2\varphi_{(\tilde{v}_1(\pm 1))_X} = \mp \frac{2\mathcal{F}_{xy}}{\mathcal{F}_{xx} - \mathcal{F}_{yy}} - \varphi_{\mathcal{P}(1,\pm 2)}$$

does not depend on X. As a consequence, we can scale the eigenvectors such that

$$(\tilde{v}_1(\pm 2))_X = (\tilde{v}_1(\pm 1))_X^2 \; .$$

This leads to $\nu_{21} = \nu_1$ and $\mp \frac{2\mathcal{F}_{xy}}{\mathcal{F}_{xx} - \mathcal{F}_{yy}} = \varphi_{\mathcal{P}(1,\pm 2)}$. Furthermore, if we define the radii ϱ_X as in the remark below lemma 3.2 (implying that $\nu_1 = 1$), then

$$\begin{aligned} |\mathcal{F}_{xx} - \mathcal{F}_{yy} \mp \mathrm{i}2\mathcal{F}_{xy}| &= \frac{8}{n}\,|\mathcal{P}(1,\pm 2)| \\ \mathcal{F}_{xx} - \mathcal{F}_{yy} \mp \mathrm{i}2\mathcal{F}_{xy} &= \frac{8}{n}\,|\mathcal{P}(1,\pm 2)| \exp\left(\mathrm{i}\varphi_{\mathcal{P}(1,\pm 2)}\right) \\ &= \frac{8}{n}\mathcal{P}(1,\pm 2) \; . \end{aligned}$$

which leads to

$$\mathcal{F}_{xx} - \mathcal{F}_{yy} = \frac{8}{n}\Re\left(\mathcal{P}(1,2)\exp\left(\mathrm{i}\phi\right)\right) = \frac{8}{n}\Re\left(\mathcal{P}(1,-2)\exp\left(-\mathrm{i}\phi\right)\right)$$

and

$$\mathcal{F}_{xy} = -\frac{4}{n}\Im\left(\mathcal{P}(1,2)\exp\left(\mathrm{i}\phi\right)\right) = \frac{4}{n}\Im\left(\mathcal{P}(1,-2)\exp\left(-\mathrm{i}\phi\right)\right)$$

Lemma 3.7 *If the scheme C^2-converges, then*

$$\left|\tilde{\lambda}_2(\pm 1)\right| < \left|\tilde{\lambda}_1(\pm 1)\right|^2$$

iff

$$\lim_{k\to\infty}\left(\frac{\left|\frac{\partial \mathcal{F}^{(k)}}{\partial x}(0,0) \mp \mathrm{i}\frac{\partial \mathcal{F}^{(k)}}{\partial y}(0,0)\right| - \left|\frac{\partial \mathcal{F}}{\partial x}(0,0) \mp \mathrm{i}\frac{\partial \mathcal{F}}{\partial y}(0,0)\right|}{\left|\tilde{\lambda}_1(\pm 1)\right|^k}\right) = 0 \; .$$

Proof From equation (9) and lemma 2.1, taking $\alpha = \alpha_X + \alpha_k$, $\Omega = 1$, and $a = (\mathcal{F}_x^{(k)})\frac{1}{\rho_X^{(k)}}$, $b = (\mathcal{F}_y^{(k)})/\rho_X^{(k)}$, we get with $\omega = \pm 1$,

$$\lim_{k\to\infty}\left(\frac{\tilde{X}^{(k)}(\pm 1) - \rho_X^{(k)}\frac{n}{2}\left(\mathcal{F}_x^{(k)} \mp i\mathcal{F}_y^{(k)}\right)\exp\left(\pm\frac{2\mathrm{i}\pi}{n}\left(\alpha_X + \alpha_k - \phi_X - \phi_k\right)\right)}{{\rho_X^{(k)}}^2}\right) = 0 \; .$$

From lemma 2.3 having supposed that $L_1(\omega) = L_2^+(\omega)$ we get

$$\tilde{X}^{(k)}(\pm 1) = \tilde{\lambda}_1(\pm 1)^k \sum_{q \in L_2^+(\pm 1)} \mathcal{P}(q, \pm 1) \left(\tilde{v}_q(\pm 1)\right)_X$$

$$+\tilde{\lambda}_2(\pm 1)^k \left(\sum_{q \in L_2(\pm 1)} \mathcal{P}(q, \pm 1) \left(\tilde{v}_q(\pm 1)\right)_X + \sum_{q \in L_2^-(\pm 1)} \left(\frac{\tilde{\lambda}_q(\pm 1)}{\tilde{\lambda}_2(\pm 1)} \right)^k \mathcal{P}(q, \pm 1) \left(\tilde{v}_q(\pm 1)\right)_X \right) .$$

So,

$$\lim_{k \to \infty} \left(\mathcal{D}_1(k) - \mathcal{D}_2(k) + \mathcal{D}_3(k)\right) = 0$$

where

$$\begin{aligned}
\mathcal{D}_1(k) &= \frac{1}{{\varrho_X^{(k)}}^2} \left(\tilde{\lambda}_1(\pm 1)^k \sum_{q \in L_2^+(\pm 1)} \mathcal{P}(q, \pm 1) \left(\tilde{v}_q(\pm 1)\right)_X \right) , \\
\mathcal{D}_2(k) &= \frac{n}{2\varrho_X^{(k)}} \left(\mathcal{F}_x^{(k)} \mp i\mathcal{F}_y^{(k)}\right) \exp\left(\pm \frac{2\mathrm{i}\pi}{n} \left(\alpha_X + \alpha_k - \phi_X - \phi_k\right) \right) , \\
\mathcal{D}_3(k) &= \frac{1}{{\varrho_X^{(k)}}^2} \left(\tilde{\lambda}_2(\pm 1)^k \left(\sum_{q \in L_2(\pm 1)} \mathcal{P}(q, \pm 1) \left(\tilde{v}_q(\pm 1)\right)_X \right.\right. \\
&\quad \left.\left. + \sum_{q \in L_2^-(\pm 1)} \left(\frac{\tilde{\lambda}_q(\pm 1)}{\tilde{\lambda}_2(\pm 1)} \right)^k \mathcal{P}(q, \pm 1) \left(\tilde{v}_q(\pm 1)\right)_X \right) \right) .
\end{aligned}$$

So,

$$\lim_{k \to \infty} \left(\mathcal{D}_3(k)\right) = 0 \quad \text{iff} \quad \lim_{k \to \infty} \left(\mathcal{D}_1(k) - \mathcal{D}_2(k)\right) = 0$$

We will prove that $\lim_{k\to\infty} \left(\mathcal{D}_1(k) - \mathcal{D}_2(k)\right) = 0$ iff

$$\lim_{k \to \infty} \left(\frac{\left(\mathcal{F}_x^{(k)} - \mathcal{F}_x\right) \pm \mathrm{i} \left(\mathcal{F}_y^{(k)} - \mathcal{F}_y\right)}{\left|\tilde{\lambda}_1(\pm 1)\right|^k} \right) = 0 .$$

From equations (7) and (8), we get

$$\mathcal{D}_1(k) = \frac{\tilde{\lambda}_1(\pm 1)^k}{\left(\left|\tilde{\lambda}_1(\pm 1)\right|^k \mathcal{A}_{X,\pm 1}(k) \right)^2} \frac{\sum_{q \in L_2^+(\pm 1)} \mathcal{P}(q, \pm 1) \left(\tilde{v}_q(\pm 1)\right)_X}{\left| \sum_{q \in L_2^+(\pm 1)} \mathcal{P}(q, \pm 1) \left(\tilde{v}_q(\pm 1)\right)_X \right|^2} \left(\frac{n}{2} \left|\mathcal{F}_x \mp \mathrm{i}\mathcal{F}_y\right| \right)^2$$

and

$$\mathcal{D}_2(k) = \frac{\mathcal{F}_x^{(k)} \mp \mathrm{i}\mathcal{F}_y^{(k)}}{\left|\tilde{\lambda}_1(\pm 1)\right|^k \mathcal{A}_{X,\pm 1}(k)} \frac{n^2}{4} \frac{\left|\mathcal{F}_x \mp \mathrm{i}\mathcal{F}_y\right|}{\left| \sum_{q \in L_2^+(\pm 1)} \mathcal{P}(q, \pm 1) \left(\tilde{v}_q(\pm 1)\right)_X \right|} \exp\left(\pm \frac{2\mathrm{i}\pi}{n} \left(\alpha_X + \alpha_k - \phi_X - \phi_k\right) \right)$$

which leads to

$$\left|\mathcal{D}_1(k)\right| - \left|\mathcal{D}_2(k)\right| = \frac{n^2}{4} \frac{\left|\mathcal{F}_x^{(k)} \mp \mathrm{i}\mathcal{F}_y^{(k)}\right|}{\left| \sum_{q \in L_2^+(\pm 1)} \mathcal{P}(q, \pm 1) \left(\tilde{v}_q(\pm 1)\right)_X \right|} \frac{\left|\mathcal{F}_x^{(k)} \mp \mathrm{i}\mathcal{F}_y^{(k)}\right| - \left|\mathcal{F}_x \mp \mathrm{i}\mathcal{F}_y\right|}{\left|\tilde{\lambda}_1(\pm 1)\right|^k \mathcal{A}_{X,\pm 1}(k)}$$

and so

$$\lim_{k\to\infty} \left(|\mathcal{D}_1(k)| - |\mathcal{D}_2(k)|\right) = 0$$

iff

$$\lim_{k\to\infty} \left(\frac{\left|\mathcal{F}_x^{(k)} \mp \mathrm{i}\mathcal{F}_y^{(k)}\right| - |\mathcal{F}_x \mp \mathrm{i}\mathcal{F}_y|}{\left|\tilde{\lambda}_1(\pm 1)\right|^k} \right) = 0 \ .$$

Furthermore,

$$\varphi_{\mathcal{D}_1(k)} = \varphi_{\tilde{\lambda}_1^k(\pm 1) \sum_{q\in L_2^+(\pm 1)} \mathcal{P}(q,\pm 1)(\tilde{v}_q(\pm 1))_X}$$

and

$$\varphi_{\mathcal{D}_2(k)} = \mp \frac{\mathcal{F}_y^{(k)}}{\mathcal{F}_x^{(k)}} \pm \frac{2\pi}{n} \left(\alpha_X + \alpha_k - \phi_X - \phi_k\right) \ .$$

If $L_1(1) = L_1(-1) = \{1\}$,

$$\varphi_{\mathcal{D}_1(k)} = k\varphi_{\tilde{\lambda}_1(\pm 1)} \varphi_{\mathcal{P}(1,\pm 1)} + \varphi_{(\tilde{v}_1(\pm 1))_X}$$

and from equations (3) and (4) we get

$$\varphi_{\mathcal{D}_2(k)} = \mp \frac{\mathcal{F}_y^{(k)}}{\mathcal{F}_x^{(k)}} + \varphi_{(\tilde{v}_1(\pm 1))_X} \pm \frac{\mathcal{F}_y}{\mathcal{F}_x} + \varphi_{\mathcal{P}(1,\pm 1)} + k\varphi_{\tilde{\lambda}_1(\pm 1)}$$

which leads to

$$\lim_{k\to\infty} \left(\varphi_{\mathcal{D}_1(k)} - \varphi_{\mathcal{D}_2(k)}\right) = 0 \ .$$

As a consequence,

$$\lim_{k\to\infty} \left(\mathcal{D}_3(k)\right) = 0$$

iff

$$\lim_{k\to\infty} \left(\frac{\left|\mathcal{F}_x^{(k)} \mp \mathrm{i}\mathcal{F}_y^{(k)}\right| - |\mathcal{F}_x \mp \mathrm{i}\mathcal{F}_y|}{\left|\tilde{\lambda}_1(\pm 1)\right|^k} \right) = 0 \ .$$

And because

$$\lim_{k\to\infty} \left(\mathcal{D}_3(k)\right) = \lim_{k\to\infty} \left(\frac{1}{{\varrho_X^{(k)}}^2} \left(\tilde{\lambda}_2(\pm 1)^k \sum_{q\in L_2(\pm 1)} \mathcal{P}(q,\pm 1) \left(\tilde{v}_q(\pm 1)\right)_X \right) \right)$$

we get

$$\left|\tilde{\lambda}_2(\pm 1)\right| < \left|\tilde{\lambda}_1(\pm 1)\right|^2$$

iff

$$\lim_{k\to\infty} \left(\frac{\left|\mathcal{F}_x^{(k)} \mp \mathrm{i}\mathcal{F}_y^{(k)}\right| - |\mathcal{F}_x \mp \mathrm{i}\mathcal{F}_y|}{\left|\tilde{\lambda}_1(\pm 1)\right|^k} \right) = 0 \ .$$

∎

Remark If the mark point is a face centre, we do not control $\mathcal{F}_x^{(k)}$ or $\mathcal{F}_y^{(k)}$. So, as explained in Sect. 2.6, the sampling may be inadequate for our analysis.

Lemma 3.8 *If the scheme C^2-converges, then for $\omega \notin \{-2,-1,0,1,2\}$,*

$$\left|\tilde{\lambda}_1(\omega)\right| < \left|\tilde{\lambda}_1(\pm 1)\right|^2 .$$

Proof From equation (9) and lemma 2.1, taking $\alpha = \alpha_X$, we get for $\omega \notin \{-2,-1,0,1,2\}$,

$$\lim_{k\to\infty}\left(\frac{\tilde{X}^{(k)}(\omega)}{{\rho_X^{(k)}}^2}\right) = 0$$

From lemma 2.3, having supposed that $L_1(\omega) = L_2^+(\omega)$ we get

$$\frac{\tilde{X}^{(k)}(\omega)}{{\rho_X^{(k)}}^2} =$$

$$\frac{\tilde{\lambda}_1(\omega)^k}{{\rho_X^{(k)}}^2}\left(\sum_{q\in L_1(\omega)} \mathcal{P}(q,\omega)\left(\tilde{v}_q(\omega)\right)_X + \sum_{q\in L_1^-(\omega)}\left(\frac{\tilde{\lambda}_q(\omega)}{\tilde{\lambda}_1(\omega)}\right)^k \mathcal{P}(q,\omega)\left(\tilde{v}_q(\omega)\right)_X\right)$$

So,

$$\begin{aligned} 0 &= \lim_{k\to\infty}\left(\frac{\tilde{\lambda}_1(\omega)^k}{{\rho_X^{(k)}}^2}\left(\sum_{q\in L_1(\omega)} \mathcal{P}(q,\omega)\left(\tilde{v}_q(\omega)\right)_X\right.\right. \\ &\quad \left.\left. + \sum_{q\in L_1^-(\omega)}\left(\frac{\tilde{\lambda}_q(\omega)}{\tilde{\lambda}_1(\omega)}\right)^k \mathcal{P}(q,\omega)\left(\tilde{v}_q(\omega)\right)_X\right)\right) \\ &= \lim_{k\to\infty}\left(\frac{\tilde{\lambda}_1(\omega)^k}{{\rho_X^{(k)}}^2}\sum_{q\in L_1(\omega)} \mathcal{P}(q,\omega)\left(\tilde{v}_q(\omega)\right)_X\right) \end{aligned}$$

which implies that,

$$\left|\tilde{\lambda}_1(\omega)\right| < \left|\tilde{\lambda}_1(\pm 1)\right|^2$$

∎

Remark The necessary conditions for C^2-convergence are quadratic domination between eigenvalues and quadratic configuration of eigenvectors. As expected, they give the values of the partial derivatives $\frac{\partial^2\mathcal{F}}{\partial x^2}(0,0)$, $\frac{\partial^2\mathcal{F}}{\partial y^2}(0,0)$ and $\frac{\partial^2\mathcal{F}}{\partial x\partial y}(0,0)$.

3.4 Discussion

Many authors interpret a subdivision scheme as a linear map between patches which progressively fill in an n-sided hole around an extraordinary point. Prautzsch [10] and Zorin [17] proposed necessary and sufficient conditions for C^p-regularity of the limit surface, on the eigenvalues and eigenbasis functions of this linear map. In contrast, we interpret a subdivision scheme as a linear map between samplings of two successive surfaces from a sequence of C^p surfaces. If this sequence converges with sufficient regularity (C^p-converges) these samplings may be used to approximate the derivatives of the limit surface. We propose necessary conditions for the C^2-convergence of a scheme, which is itself a sufficient condition for the C^2-continuity of the limit surface, on the eigenvalues and eigenvectors of the transformed subdivision matrix. As already stated, a scheme which converges toward a C^2-continuous limit surface does not necessarily C^2-converge. But it is interesting to understand the difference between our necessary conditions for C^p-convergence, and the condition for the C^p-regularity of the limit surface proposed by Reif, Prautzsch and Zorin.

C^0-regularity We find the same conditions.

C^1-regularity Because we ask the sub-dominant eigenvalues to come from $\tilde{\mathrm{M}}(1)$ and $\tilde{\mathrm{M}}(-1)$, we assure the orthoradial injectivity of Reif's characteristic map as described in [9]; and because we ask the components of the associated eigenvectors to be sorted like the parameters $\varrho_X^{(k)}$, we assure the radial injectivity of this map.

C^2-regularity Reif's characteristic map [12] is given by the sub-dominant eigenbasis functions. If the scheme is Box-Spline based, the eigenbasis functions are Box-Splines with our eigenvectors as control points (more precisely, our eigenvectors provide their radial coordinates). One of the conditions proposed by Prautzsch [10] and Zorin [17] for C^2-regularity, is that the eigenbasis functions z associated with the sub-sub-dominant eigenvalue should belong to span $\{x^i y^j; i+j=2\}$ where x and y are the eigenbasis functions associated with the sub-dominant eigenvalue. Our condition is the same, but with the eigenvectors instead of the eigenbasis functions. And the eigenvectors provide the altitude over the characteristic map of the control points of z. Around an ordinary vertex, we have checked that the quadratic configuration of the eigenvectors is fulfilled for the Loop and Catmull-Clark schemes. Stam does this for the quadratic configuration of eigenbasis functions [16]. The possibility of getting quadratic configuration of both eigenvectors and eigenbasis functions around an extraordinary vertex remains to be investigated.

4 Converting the Analysis to the Real Domain

The necessary conditions for C^2-convergence of a scheme, proposed in the previous section, concern the eigenvalues and eigenvectors of subdivision matrices in the frequency domain. The components of these matrices may be complex. Having them real would simplify numerical analysis of the eigenstructure of the matrices, especially in the context of scheme tuning where we manipulate symbolic terms.

In this section, we present some mechanisms to make the subdivision matrices in the frequency domain $\tilde{\mathrm{M}}(\omega)$ real. We will prove that choosing convenient phases in the parameter space makes some of these components real, but an additional mechanism is necessary to make all of them real: vertex substitution. We derive necessary conditions on these new real matrices for C^2-convergence of the scheme.

4.1 Choosing Convenient Phases

Lemma 4.1 *If the scheme has rotational and p-mirror invariances, then for all* $X \in \{B, C, D, \ldots\}$*, for all* $Y \in \{B, C, D, \ldots\}$*, the components* $\tilde{M}_{l_X,l_Y}(\omega)$ *are real if* X *or* Y *do not belong to the floor,* $d\phi = p/2$*, and the difference between the phases* $\phi_X - \phi_Y$ *is chosen as follows:*

$X \backslash Y$	*basement*	*ceiling*
basement	0	$-1/2$
ceiling	$1/2$	0

Furthermore, if X *or* Y *belongs to the floor, no phase makes the component* $\tilde{M}_{l_X,l_Y}(\omega)$ *real.*

Proof From lemma 2.4,

$$\tilde{M}_{l_X,l_Y}(\omega) = \sum_{q=1}^{n} m_{(X,Y),q} \exp\left(-\frac{2i\pi\omega}{n}(q + \phi_X - \phi_Y + d\phi)\right) .$$

If X and Y belong to the basement then, from lemma 2.6,

$$m_{(X,Y),q} = m_{(X,Y),-q-p} .$$

So, if $n - p - 1$ is even,

$$\begin{aligned}
\tilde{M}_{l_X,l_Y}(\omega) &= \sum_{q=n-p}^{n} m_{(X,Y),q} \exp\left(-\frac{2i\pi\omega}{n}(q + \phi_X - \phi_Y + d\phi)\right) \\
&+ \sum_{q=1}^{\frac{n-1-p}{2}} m_{(X,Y),q} \exp\left(-\frac{2i\pi\omega}{n}(q + \phi_X - \phi_Y + d\phi)\right) \\
&+ \sum_{q=\frac{n-1-p}{2}+1}^{n-1-p} m_{(X,Y),q} \exp\left(-\frac{2i\pi\omega}{n}(q + \phi_X - \phi_Y + d\phi)\right) \\
&= \sum_{q=n-p}^{n} m_{(X,Y),q} \exp\left(-\frac{2i\pi\omega}{n}(q + \phi_X - \phi_Y + d\phi)\right) \\
&+ \sum_{q=1}^{\frac{n-1-p}{2}} m_{(X,Y),q} \exp\left(-\frac{2i\pi\omega}{n}(q + \phi_X - \phi_Y + d\phi)\right) \\
&+ \sum_{q=1}^{\frac{n-1-p}{2}} m_{(X,Y),n-q-p} \exp\left(-\frac{2i\pi\omega}{n}(n - q - p + \phi_X - \phi_Y + d\phi)\right) \\
&= \sum_{q=n-p}^{n} m_{(X,Y),q} \exp\left(-\frac{2i\pi\omega}{n}(q + \phi_X - \phi_Y + d\phi)\right)
\end{aligned}$$

$$+ \sum_{q=1}^{\frac{n-1-p}{2}} m_{(X,Y),q}$$

$$\left(\exp\left(-\frac{2i\pi\omega}{n}(q+\phi_X-\phi_Y+d\phi)\right)+\exp\left(-\frac{2i\pi\omega}{n}(n-q-p+\phi_X-\phi_Y+d\phi)\right)\right)$$

The last sum is real if (all the equalities are modulo n),

$$\begin{aligned} q+\phi_X-\phi_Y+d\phi &= -(-q-p+\phi_X-\phi_Y+d\phi) \\ 2(\phi_X-\phi_Y+d\phi) &= p \end{aligned}$$

In particular, this has to be true if $X=Y$. So,

$$d\phi = \frac{p}{2}$$

and

$$\phi_X = \phi_Y.$$

The first sum remains. If p is even, with the relation derived above,

$$\begin{aligned} &\sum_{q=n-p}^{n} m_{(X,Y),q} \exp\left(-\frac{2i\pi\omega}{n}(q+\phi_X-\phi_Y+d\phi)\right) \\ &= m_{(X,Y),n-(p/2)} \exp\left(-\frac{2i\pi\omega}{n}(n-(p/2)+\phi_X-\phi_Y+d\phi)\right) \\ &+ \sum_{q=n-p}^{n-p+(p/2)-1} m_{(X,Y),q} \exp\left(-\frac{2i\pi\omega}{n}(q+\phi_X-\phi_Y+d\phi)\right) \\ &+ \sum_{q=n-p+(p/2)+1}^{n} m_{(X,Y),q} \exp\left(-\frac{2i\pi\omega}{n}(q+\phi_X-\phi_Y+d\phi)\right) \\ &= m_{(X,Y),n-(p/2)} \\ &+ \sum_{q=n-p}^{n-p+(p/2)-1} m_{(X,Y),q} \\ &\left(\exp\left(-\frac{2i\pi\omega}{n}(q+\frac{p}{2})\right)+\exp\left(-\frac{2i\pi\omega}{n}(n-q-p+\frac{p}{2})\right)\right) \\ &= m_{(X,Y),n-(p/2)} + \sum_{q=n-p}^{n-(p/2)-1} m_{(X,Y),q} 2\cos\left(\frac{2\pi\omega}{n}(q+\frac{p}{2})\right) \end{aligned}$$

Finally, if $n-p-1$ is even and p even,

$$\begin{aligned} \tilde{M}_{l_X,l_Y}(\omega) &= m_{(X,Y),n-(p/2)} + \sum_{q=n-p}^{n-(p/2)-1} m_{(X,Y),q} 2\cos\left(\frac{2\pi\omega}{n}(q+\frac{p}{2})\right) \\ &+ \sum_{q=1}^{\frac{n-1-p}{2}} m_{(X,Y),q} 2\cos\left(\frac{2\pi\omega}{n}(q+\frac{p}{2})\right) \\ &= m_{(X,Y),n-(p/2)} + \sum_{q=1}^{n-(p/2)-1} m_{(X,Y),q} 2\cos\left(\frac{2\pi\omega}{n}(q+\frac{p}{2})\right) \end{aligned}$$

If p is odd, with the relation derived above,

$$\sum_{q=n-p}^{n} m_{(X,Y),q} \exp\left(-\frac{2i\pi\omega}{n}(q+\phi_X-\phi_Y+d\phi)\right)$$
$$= \sum_{q=n-p}^{n-p+(p+1)/2-1} m_{(X,Y),q} \exp\left(-\frac{2i\pi\omega}{n}(q+\phi_X-\phi_Y+d\phi)\right)$$
$$+ \sum_{q=n-p+(p+1)/2}^{n} m_{(X,Y),q} \exp\left(-\frac{2i\pi\omega}{n}(q+\phi_X-\phi_Y+d\phi)\right)$$
$$= \sum_{q=n-p}^{n-(p+1)/2)} m_{(X,Y),q}$$
$$\left(\exp\left(-\frac{2i\pi\omega}{n}(q+\frac{p}{2})\right) + \exp\left(-\frac{2i\pi\omega}{n}(n-q-p+\frac{p}{2})\right)\right)$$
$$= \sum_{q=n-p}^{n-(p+1)/2} m_{(X,Y),q} 2\cos\left(\frac{2\pi\omega}{n}(q+\frac{p}{2})\right)$$

Finally, if $n-p-1$ is even and p odd,

$$\begin{aligned}\tilde{M}_{l_X,l_Y}(\omega) &= \sum_{q=n-p}^{n-(p+1)/2} m_{(X,Y),q} 2\cos\left(\frac{2\pi\omega}{n}(q+\frac{p}{2})\right) \\ &\quad + \sum_{q=1}^{\frac{n-1-p}{2}} m_{(X,Y),q} 2\cos\left(\frac{2\pi\omega}{n}(q+\frac{p}{2})\right) \\ &= \sum_{q=1}^{n-(p+1)/2} m_{(X,Y),q} 2\cos\left(\frac{2\pi\omega}{n}(q+\frac{p}{2})\right)\end{aligned}$$

If $n-p-1$ is odd,

$$\tilde{M}_{l_X,l_Y}(\omega) = \sum_{q=n-p}^{n} m_{(X,Y),q} \exp\left(-\frac{2i\pi\omega}{n}(q+\phi_X-\phi_Y+d\phi)\right)$$
$$m_{(X,Y),(n-p)/2} \exp\left(-\frac{2i\pi\omega}{n}((n-p)/2+\phi_X-\phi_Y+d\phi)\right)$$
$$+ \sum_{q=1}^{(n-p)/2-1} m_{(X,Y),q} \exp\left(-\frac{2i\pi\omega}{n}(q+\phi_X-\phi_Y+d\phi)\right)$$
$$+ \sum_{q=(n-p)/2+1}^{n-1-p} m_{(X,Y),q} \exp\left(-\frac{2i\pi\omega}{n}(q+\phi_X-\phi_Y+d\phi)\right)$$
$$= \sum_{q=n-p}^{n} m_{(X,Y),q} \exp\left(-\frac{2i\pi\omega}{n}(q+\phi_X-\phi_Y+d\phi)\right)$$
$$m_{(X,Y),(n-p)/2} \exp\left(-\frac{2i\pi\omega}{n}((n-p)/2+\phi_X-\phi_Y+d\phi)\right)$$

$$+ \sum_{q=1}^{(n-p)/2-1} m_{(X,Y),q} \exp\left(-\frac{2\mathrm{i}\pi\omega}{n}(q+\phi_X-\phi_Y+d\phi)\right)$$
$$+ \sum_{q=1}^{(n-p)/2-1} m_{(X,Y),n-q-p} \exp\left(-\frac{2\mathrm{i}\pi\omega}{n}(n-q-p+\phi_X-\phi_Y+d\phi)\right)$$
$$= \sum_{q=n-p}^{n} m_{(X,Y),q} \exp\left(-\frac{2\mathrm{i}\pi\omega}{n}(q+\phi_X-\phi_Y+d\phi)\right)$$
$$m_{(X,Y),(n-p)/2} \exp\left(-\frac{2\mathrm{i}\pi\omega}{n}((n-p)/2+\phi_X-\phi_Y+d\phi)\right)$$
$$+ \sum_{q=1}^{(n-p)/2-1} m_{(X,Y),q}$$
$$\left(\exp\left(-\frac{2\mathrm{i}\pi\omega}{n}(q+\phi_X-\phi_Y+d\phi)\right) + \exp\left(-\frac{2\mathrm{i}\pi\omega}{n}(n-q-p+\phi_X-\phi_Y+d\phi)\right)\right)$$

As before, the last sum is real if

$$d\phi = \frac{p}{2}$$

and

$$\phi_X = \phi_Y \ ,$$

and the first sum is then real. There remains the following term

$$m_{(X,Y),(n-p)/2} \exp\left(-\frac{2\mathrm{i}\pi\omega}{n}((n-p)/2+\phi_X-\phi_Y+d\phi)\right)$$
$$= m_{(X,Y),(n-p)/2} \exp\left(-\frac{2\mathrm{i}\pi\omega}{n}((n-p)/2+p/2)\right)$$
$$= m_{(X,Y),(n-p)/2} \exp\left(-\mathrm{i}\pi\omega\right)$$

which is real as well.

The same proof is valid if X and Y belong to the ceiling.

If X belongs to the basement and Y to the ceiling then, from lemma 2.6,

$$m_{(X,Y),q} = m_{(X,Y),1-q-p} \ .$$

So, if $n-p$ is even,

$$\tilde{M}_{l_X,l_Y}(\omega) = \sum_{q=n-p+1}^{n} m_{(X,Y),q} \exp\left(-\frac{2\mathrm{i}\pi\omega}{n}(q+\phi_X-\phi_Y+d\phi)\right)$$
$$+ \sum_{q=1}^{\frac{n-p}{2}} m_{(X,Y),q} \exp\left(-\frac{2\mathrm{i}\pi\omega}{n}(q+\phi_X-\phi_Y+d\phi)\right)$$
$$+ \sum_{q=\frac{n-p}{2}+1}^{n-p} m_{(X,Y),q} \exp\left(-\frac{2\mathrm{i}\pi\omega}{n}(q+\phi_X-\phi_Y+d\phi)\right)$$

$$= \sum_{q=n-p+1}^{n} m_{(X,Y),q} \exp\left(-\frac{2\mathrm{i}\pi\omega}{n}(q+\phi_X-\phi_Y+d\phi)\right)$$

$$+\sum_{q=1}^{\frac{n-p}{2}} m_{(X,Y),q} \exp\left(-\frac{2\mathrm{i}\pi\omega}{n}(q+\phi_X-\phi_Y+d\phi)\right)$$

$$+\sum_{q=1}^{\frac{n-p}{2}} m_{(X,Y),n+1-q-p} \exp\left(-\frac{2\mathrm{i}\pi\omega}{n}(n+1-q-p+\phi_X-\phi_Y+d\phi)\right)$$

$$= \sum_{q=n-p+1}^{n} m_{(X,Y),q} \exp\left(-\frac{2\mathrm{i}\pi\omega}{n}(q+\phi_X-\phi_Y+d\phi)\right)$$

$$+\sum_{q=1}^{\frac{n-p}{2}} m_{(X,Y),q}$$

$$\left(\exp\left(-\frac{2\mathrm{i}\pi\omega}{n}(q+\phi_X-\phi_Y+d\phi)\right)+\exp\left(-\frac{2\mathrm{i}\pi\omega}{n}(n+1-q-p+\phi_X-\phi_Y+d\phi)\right)\right)$$

The last sum is real if (all the equalities are modulo n),

$$\begin{aligned} q+\phi_X-\phi_Y+d\phi &= -(-q+1-p+\phi_X-\phi_Y+d\phi) \\ 2(\phi_X-\phi_Y+d\phi) &= -1+p \end{aligned}$$

In particular, we know from above that

$$d\phi = \frac{p}{2}$$

So,

$$\phi_X-\phi_Y = -1/2.$$

The first sum remains. If $p-1$ is even, with the relation derived above,

$$\sum_{q=n-p+1}^{n} m_{(X,Y),q} \exp\left(-\frac{2\mathrm{i}\pi\omega}{n}(q+\phi_X-\phi_Y+d\phi)\right)$$

$$= m_{(X,Y),n-(p-1)/2} \exp\left(-\frac{2\mathrm{i}\pi\omega}{n}(n-(p-1)/2+\phi_X-\phi_Y+d\phi)\right)$$

$$+ \sum_{q=n-p+1}^{n-p+(p-1)/2-1} m_{(X,Y),q} \exp\left(-\frac{2\mathrm{i}\pi\omega}{n}(q+\phi_X-\phi_Y+d\phi)\right)$$

$$+ \sum_{q=n-p+(p-1)/2+1}^{n} m_{(X,Y),q} \exp\left(-\frac{2\mathrm{i}\pi\omega}{n}(q+\phi_X-\phi_Y+d\phi)\right)$$

$$= m_{(X,Y),n-(p-1)/2}$$

$$+ \sum_{q=n-p+1}^{n-p+(p-1)/2-1} m_{(X,Y),q}$$

$$\left(\exp\left(-\frac{2\mathrm{i}\pi\omega}{n}(q-1/2+\frac{p}{2})\right)+\exp\left(-\frac{2\mathrm{i}\pi\omega}{n}(n+1-q-p-1/2+\frac{p}{2})\right)\right)$$

$$= m_{(X,Y),n-(p-1)/2} + \sum_{q=n-p+1}^{n-p+(p-1)/2-1} m_{(X,Y),q} 2\cos\left(\frac{2\pi\omega}{n}(q-1/2+\frac{p}{2})\right)$$

Finally, if $n-p$ is even and $p-1$ even,

$$\begin{aligned}
\tilde{M}_{l_X,l_Y}(\omega) &= m_{(X,Y),n-(p-1)/2} + \sum_{q=n-p+1}^{n-p+(p-1)/2-1} m_{(X,Y),q} 2\cos\left(\frac{2\pi\omega}{n}(q-1/2+\frac{p}{2})\right) \\
&\quad + \sum_{q=1}^{\frac{n-p}{2}} m_{(X,Y),q} 2\cos\left(\frac{2\pi\omega}{n}(q-1/2+p/2)\right) \\
&= m_{(X,Y),n-(p-1)/2} + \sum_{q=1}^{n-p+(p-1)/2-1} m_{(X,Y),q} 2\cos\left(\frac{2\pi\omega}{n}(q-1/2+p/2)\right)
\end{aligned}$$

If $p-1$ is odd, then in a similar fashion to that used above, we can prove that the sum is real. Finally, if $n-p$ is even and $p-1$ odd, $\tilde{M}_{l_X,l_Y}(\omega)$ is real.

If $n-p$ is odd,

$$\begin{aligned}
\tilde{M}_{l_X,l_Y}(\omega) &= \sum_{q=n-p+1}^{n} m_{(X,Y),q} \exp\left(-\frac{2\mathrm{i}\pi\omega}{n}(q+\phi_X-\phi_Y+d\phi)\right) \\
&+ m_{(X,Y),(n-p+1)/2} \exp\left(-\frac{2\mathrm{i}\pi\omega}{n}((n-p+1)/2+\phi_X-\phi_Y+d\phi)\right) \\
&+ \sum_{q=1}^{\frac{n-1-p}{2}} m_{(X,Y),q} \exp\left(-\frac{2\mathrm{i}\pi\omega}{n}(q+\phi_X-\phi_Y+d\phi)\right) \\
&+ \sum_{q=\frac{n-1-p}{2}+2}^{n-p} m_{(X,Y),q} \exp\left(-\frac{2\mathrm{i}\pi\omega}{n}(q+\phi_X-\phi_Y+d\phi)\right) \\
&= \sum_{q=n-p+1}^{n} m_{(X,Y),q} \exp\left(-\frac{2\mathrm{i}\pi\omega}{n}(q+\phi_X-\phi_Y+d\phi)\right) \\
&+ m_{(X,Y),(n-p+1)/2} \exp\left(-\mathrm{i}\pi\omega\right) \\
&+ \sum_{q=1}^{\frac{n-1-p}{2}} m_{(X,Y),q} \exp\left(-\frac{2\mathrm{i}\pi\omega}{n}(q+\phi_X-\phi_Y+d\phi)\right) \\
&+ \sum_{q=1}^{\frac{n-1-p}{2}} m_{(X,Y),n+1-q-p} \exp\left(-\frac{2\mathrm{i}\pi\omega}{n}(n+1-q-p+\phi_X-\phi_Y+d\phi)\right) \\
&= \sum_{q=n-p+1}^{n} m_{(X,Y),q} \exp\left(-\frac{2\mathrm{i}\pi\omega}{n}(q+\phi_X-\phi_Y+d\phi)\right) \\
&+ m_{(X,Y),(n-p+1)/2} \exp\left(-\mathrm{i}\pi\omega\right) \\
&+ \sum_{q=1}^{\frac{n-1-p}{2}} m_{(X,Y),q}
\end{aligned}$$

$$\left(\exp\left(-\frac{2i\pi\omega}{n}(q-1/2+p/2)\right)+\exp\left(-\frac{2i\pi\omega}{n}(n+1/2-q-p/2)\right)\right)$$

which is real.

If X belongs to the ceiling and Y to the basement then, from lemma 2.6,

$$m_{(X,Y),q} = m_{(X,Y),-1-q-p} \,.$$

So, if $n-p-2$ is even,

$$\begin{aligned}
\tilde{M}_{l_X,l_Y}(\omega) = &\sum_{q=n-p-1}^{n} m_{(X,Y),q}\exp\left(-\frac{2i\pi\omega}{n}(q+\phi_X-\phi_Y+d\phi)\right)\\
+&\sum_{q=1}^{\frac{n-p-2}{2}} m_{(X,Y),q}\exp\left(-\frac{2i\pi\omega}{n}(q+\phi_X-\phi_Y+d\phi)\right)\\
+&\sum_{q=\frac{n-p-2}{2}+1}^{n-p-2} m_{(X,Y),q}\exp\left(-\frac{2i\pi\omega}{n}(q+\phi_X-\phi_Y+d\phi)\right)\\
=&\sum_{q=n-p-1}^{n} m_{(X,Y),q}\exp\left(-\frac{2i\pi\omega}{n}(q+\phi_X-\phi_Y+d\phi)\right)\\
+&\sum_{q=1}^{\frac{n-p-2}{2}} m_{(X,Y),q}\exp\left(-\frac{2i\pi\omega}{n}(q+\phi_X-\phi_Y+d\phi)\right)\\
+&\sum_{q=1}^{\frac{n-p-2}{2}} m_{(X,Y),n-1-q-p}\exp\left(-\frac{2i\pi\omega}{n}(n-1-q-p+\phi_X-\phi_Y+d\phi)\right)\\
=&\sum_{q=n-p-1}^{n} m_{(X,Y),q}\exp\left(-\frac{2i\pi\omega}{n}(q+\phi_X-\phi_Y+d\phi)\right)\\
+&\sum_{q=1}^{\frac{n-p-2}{2}} m_{(X,Y),q}\\
&\left(\exp\left(-\frac{2i\pi\omega}{n}(q+\phi_X-\phi_Y+d\phi)\right)+\exp\left(-\frac{2i\pi\omega}{n}(n-1-q-p+\phi_X-\phi_Y+d\phi)\right)\right)
\end{aligned}$$

The last sum is real if (all the equalities are modulo n),

$$\begin{aligned}
q+\phi_X-\phi_Y+d\phi &= -(-q-1-p+\phi_X-\phi_Y+d\phi)\\
2(\phi_X-\phi_Y+d\phi) &= 1+p
\end{aligned}$$

In particular, we know from above that

$$d\phi = \frac{p}{2}$$

So,

$$\phi_X-\phi_Y = 1/2.$$

The first sum remains. If $p+1$ is even, with the relation derived above,

$$\sum_{q=n-p-1}^{n} m_{(X,Y),q} \exp\left(-\frac{2\mathrm{i}\pi\omega}{n}(q+\phi_X-\phi_Y+d\phi)\right)$$
$$= m_{(X,Y),n-(p+1)/2} \exp\left(-\frac{2\mathrm{i}\pi\omega}{n}(n-(p+1)/2+\phi_X-\phi_Y+d\phi)\right)$$
$$+ \sum_{q=n-p-1}^{n-p+(p+1)/2-1} m_{(X,Y),q} \exp\left(-\frac{2\mathrm{i}\pi\omega}{n}(q+\phi_X-\phi_Y+d\phi)\right)$$
$$+ \sum_{q=n-p+(p+1)/2+1}^{n} m_{(X,Y),q} \exp\left(-\frac{2\mathrm{i}\pi\omega}{n}(q+\phi_X-\phi_Y+d\phi)\right)$$
$$= m_{(X,Y),n-(p+1)/2}$$
$$+ \sum_{q=n-p-1}^{n-p+(p+1)/2-1} m_{(X,Y),q}$$
$$\left(\exp\left(-\frac{2\mathrm{i}\pi\omega}{n}(q+1/2+\frac{p}{2})\right) + \exp\left(-\frac{2\mathrm{i}\pi\omega}{n}(n-1-q-p+1/2+\frac{p}{2})\right)\right)$$
$$= m_{(X,Y),n-(p+1)/2} + \sum_{q=n-p-1}^{n-p+(p+1)/2-1} m_{(X,Y),q} 2\cos\left(\frac{2\pi\omega}{n}(q+1/2+\frac{p}{2})\right)$$

Finally, if $n-p-2$ is even and $p+1$ even,

$$\begin{aligned}
\tilde{M}_{l_X,l_Y}(\omega) &= m_{(X,Y),n-(p+1)/2} + \sum_{q=n-p-1}^{n-p+(p+1)/2-1} m_{(X,Y),q} 2\cos\left(\frac{2\pi\omega}{n}(q+1/2+\frac{p}{2})\right) \\
&\quad + \sum_{q=1}^{\frac{n-2-p}{2}} m_{(X,Y),q} 2\cos\left(\frac{2\pi\omega}{n}(q+1/2+p/2)\right) \\
&= m_{(X,Y),n-(p+1)/2} + \sum_{q=1}^{n-p+(p+1)/2-1} m_{(X,Y),q} 2\cos\left(\frac{2\pi\omega}{n}(q+1/2+p/2)\right)
\end{aligned}$$

If $p+1$ is odd, in a similar fashion to that used above, we can prove that the sum is real. Finally, if $n-p-2$ is even and $p+1$ odd, $\tilde{M}_{l_X,l_Y}(\omega)$ is real.

If $n-p-2$ is odd,

$$\tilde{M}_{l_X,l_Y}(\omega) = \sum_{q=n-p-1}^{n} m_{(X,Y),q} \exp\left(-\frac{2\mathrm{i}\pi\omega}{n}(q+\phi_X-\phi_Y+d\phi)\right)$$
$$+ m_{(X,Y),(n-p-1)/2} \exp\left(-\frac{2\mathrm{i}\pi\omega}{n}((n-p-1)/2+\phi_X-\phi_Y+d\phi)\right)$$
$$+ \sum_{q=1}^{\frac{n-1-p}{2}-1} m_{(X,Y),q} \exp\left(-\frac{2\mathrm{i}\pi\omega}{n}(q+\phi_X-\phi_Y+d\phi)\right)$$
$$+ \sum_{q=\frac{n-1-p}{2}+1}^{n-2-p} m_{(X,Y),q} \exp\left(-\frac{2\mathrm{i}\pi\omega}{n}(q+\phi_X-\phi_Y+d\phi)\right)$$

$$= \sum_{q=n-p-1}^{n} m_{(X,Y),q} \exp\left(-\frac{2i\pi\omega}{n}(q+\phi_X-\phi_Y+d\phi)\right)$$
$$+m_{(X,Y),(n-p-1)/2} \exp(-i\pi\omega)$$
$$+ \sum_{q=1}^{\frac{n-1-p}{2}-1} m_{(X,Y),q} \exp\left(-\frac{2i\pi\omega}{n}(q+\phi_X-\phi_Y+d\phi)\right)$$
$$+ \sum_{q=1}^{\frac{n-1-p}{2}-1} m_{(X,Y),n-1-q-p} \exp\left(-\frac{2i\pi\omega}{n}(n-1-q-p+\phi_X-\phi_Y+d\phi)\right)$$
$$= \sum_{q=n-p-1}^{n} m_{(X,Y),q} \exp\left(-\frac{2i\pi\omega}{n}(q+\phi_X-\phi_Y+d\phi)\right)$$
$$+m_{(X,Y),(n-p-1)/2} \exp(-i\pi\omega)$$
$$+ \sum_{q=1}^{\frac{n-1-p}{2}-1} m_{(X,Y),q}$$
$$\left(\exp\left(-\frac{2i\pi\omega}{n}(q+1/2+p/2)\right) + \exp\left(-\frac{2i\pi\omega}{n}(n-1/2-q-p/2)\right)\right)$$

which is real.

If X or Y belongs to floor then, from lemma 2.6, we get relationship between $m_{(X,Y),*}$ and $m_{(X',Y),*}$ or $m_{(X,Y'),*}$ or $m_{(X',Y'),*}$ which does not lead to any simplification in the writing of $\tilde{M}_{l_X,l_Y}(\omega)$. ∎

Remark This lemma tells us which formula of the discrete Fourier transform we have to choose for each vertex when we write the subdivision matrices in the frequency domain. In practice, this is equivalent to giving at each spatial vertex an index which is equal to its angular coordinate (divided by $2\pi/n$) in the parameter space and then applying the classical formula of the discrete Fourier transform (without phases). Note that if p is not null, the grid of the new vertices must be rotated by $2p\pi/n$ and their angular coordinates, which will provide the indices, have to be taken in the parameter space corresponding to the old vertices. For an application of this technique see [2].

4.2 Substituting Vertices

Lemma 4.1 tells us that if X or Y belongs to the floor, no phase makes the component $\tilde{M}_{l_X,l_Y}(\omega)$ real. To overcome this problem, the solution is to exchange the vertices which belong to the floor with new vertices defined as linear combinations of themselves with their mirror images. The linear combination is chosen such that these new vertices belong to the basement or the ceiling, as shown in Fig. 3. In this section we will refer to the initial vertices as old vertices and to the vertices defined as a linear combination of an old vertex which belongs to the floor with its mirror image as new vertices.

Lemma 4.2 *If the scheme is rotationally and mirror invariant, and if, for all X belonging to the floor, we substitute in the vector $\mathbf{P}^{(k)}$ the vertices X_j and X'_j by*

$$H_j^{(k)} = \frac{X_j^{(k)} + X'^{(k)}_{j-1}}{2} \quad \textit{and} \quad I_j^{(k)} = \frac{X_j^{(k)} + X'^{(k)}_j}{2}\ ,$$

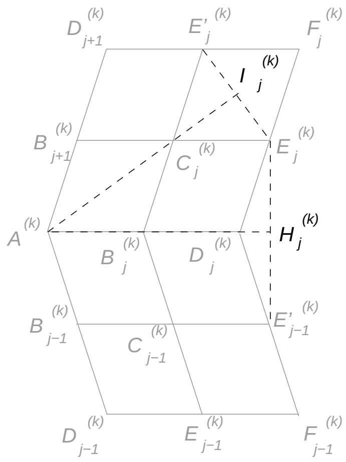


Figure 3: Definition on new points lying on symmetry axes

then the new components of the subdivision matrix M *may be written, if the mark point is a vertex, as*

$$\begin{aligned} m_{H,1} &= (m_{X,1} + m_{X',1})/2 \ , \\ m_{I,1} &= (m_{X,1} + x_{X',1})/2 \ . \end{aligned}$$

and, whatever the mark point is, for all old Y belonging to the basement or to the ceiling, for all $q \in \{1 \dots n\}$, as

$$\begin{aligned} m_{(H,Y),q} &= (m_{(X,Y),q} + m_{(X',Y),q-1})/2 \ , \\ m_{(I,Y),q} &= (m_{(X,Y),q} + m_{(X',Y),q})/2 \ . \end{aligned}$$

Proof The new vertices $H_j^{(k)}$ are defined as

$$H_j^{(k)} = \frac{X_j^{(k)} + X'^{(k)}_{j-1}}{2} \ .$$

So, as in the proof of lemma 2.4,

$$\begin{aligned} H_j^{(k+1)} &= \frac{1}{2}\Bigg(m_{X,1} A^{(k)} + \sum_{Y \in B,C,D,\dots} \sum_{q=1}^{n} m_{(X,Y),q} Y_{j-q}^{(k)} \\ &\quad + m_{X',1} A^{(k)} + \sum_{Y \in B,C,D,\dots} \sum_{q=1}^{n} m_{(X',Y),q} Y_{j-1-q}^{(k)} \Bigg) \ . \end{aligned}$$

which leads to

$$m_{H,1} = (m_{X,1} + x_{X',1})/2 \ .$$

Furthermore,

$$\sum_{q=1}^{n} m_{(X',Y),q} Y_{j-1-q}^{(k)} \quad = \quad \sum_{q=2}^{n+1} m_{(X',Y),q-1} Y_{j-q}^{(k)}$$

$$= \sum_{q=1}^{n} m_{(X',Y),q-1} Y_{j-q}^{(k)} + m_{(X',Y),n+1-1} Y_{j-n-1}^{(k)} - m_{(X',Y),1-1} Y_{j-1}^{(k)}$$
$$= \sum_{q=1}^{n} m_{(X',Y),q-1} Y_{j-q}^{(k)} .$$

So,

$$m_{(H,Y),q} = (m_{(X,Y),q} + m_{(X',Y),q-1})/2 .$$

Similarly, we prove

$$m_{I,1} = (m_{X,1} + x_{X',1})/2 \quad \text{and} \quad m_{(I,Y),q} = (m_{(X,Y),q} + m_{(X',Y),q})/2$$

∎

Lemma 4.3 *If the scheme is rotationally and mirror invariant, for all X belonging to the floor, we substitute the vertices X_j and X'_j in the vector $\mathbf{P}^{(k)}$ by*

$$H_j^{(k)} = \frac{X_j^{(k)} + X'^{(k)}_{j-1}}{2} \quad \textit{and} \quad I_j^{(k)} = \frac{X_j^{(k)} + X'^{(k)}_{j}}{2} .$$

If the mark point is a vertex, then

$$m_{1,I} = 2m_{1,X} ,$$
$$m_{1,H} = 0 .$$

For simplicity, for every old vertex Y which belongs to the floor, and for every new vertices H and I coming from old E and E', we formally define the following coefficients which are not new components of the subdivision matrix M*:*

$$m_{(H,Y),q} = (m_{(X,Y),q} + m_{(X',Y),q-1})/2 ,$$
$$m_{(I,Y),q} = (m_{(X,Y),q} + m_{(X',Y),q})/2 .$$

Then, whatever the mark point is, the new components of the subdivision matrix M *may be written for all Y being an old vertex belonging to the basement or a new vertex being H-like, and for all $q \in \{1 \dots n\}$, as*

$$m_{(Y,H),q} = -2 \left(\sum_{l=q+1}^{n} m_{(Y,X),l} + \sum_{l=n-q+1}^{n} m_{(Y,X),l} \right)$$
$$m_{(Y,I),q} = 2 \left(\sum_{l=q}^{n} m_{(Y,X),l} + \sum_{l=n-q+1}^{n} m_{(Y,X),l} \right) .$$

Furthermore, the new components of the subdivision matrix M *may be written for all Y being an old vertex belonging to the ceiling, or a new vertex which is I-like, and for all $q \in \{1 \dots n\}$, as*

$$m_{(Y,H),q} = -2 \left(\sum_{l=q+1}^{n} m_{(Y,X),l} + \sum_{l=n-q}^{n} m_{(Y,X),l} \right)$$
$$m_{(Y,I),q} = 2 \left(\sum_{l=q}^{n} m_{(Y,X),l} + \sum_{l=n-q}^{n} m_{(Y,X),l} \right) .$$

Proof If the mark point is a vertex, then, as in the proof of lemma 2.4,

$$A^{(k+1)} = M_{1,1}A^{(k)} + \sum_{Z\notin\{X,X'\}} \sum_{q=1}^{n} m_{1,Z} Z^{(k)}_{j-q} + \sum_{q=1}^{n} m_{1,X} X^{(k)}_{j-q} + \sum_{q=1}^{n} m_{1,X'} X'^{(k)}_{j-q} \ .$$

The scheme being mirror invariant,

$$m_{1,X} = m_{1,X'}$$

So,

$$\begin{aligned} A^{(k+1)} &= M_{1,1}A^{(k)} + \sum_{Z\notin\{X,X'\}} \sum_{q=1}^{n} m_{1,Z} Z^{(k)}_{j-q} + \sum_{q=1}^{n} m_{1,X} \left(X^{(k)}_{j-q} + X'^{(k)}_{j-q} \right) \\ &= M_{1,1}A^{(k)} + \sum_{Z\notin\{X,X'\}} \sum_{q=1}^{n} m_{1,Z} Z^{(k)}_{j-q} + \sum_{q=1}^{n} 2 m_{1,X} I^{(k)}_{j-q} \ . \end{aligned}$$

Let Y be an old vertex which belongs to the basement. As in the proof of lemma 2.4,

$$Y_j^{(k+1)} = m_{Y,1}A^{(k)} + \sum_{Z\notin\{X,X'\}} \sum_{q=1}^{n} m_{(Y,Z),q} Z^{(k)}_{j-q} + \sum_{q=1}^{n} m_{(Y,X),q} X^{(k)}_{j-q} + \sum_{q=1}^{n} m_{(Y,X'),q} X'^{(k)}_{j-q} \ .$$

And, from lemma 2.6,

$$m_{(Y,X),q} = m_{(Y,X'),1-q}$$

So,

$$\begin{aligned} \sum_{q=1}^{n} m_{(Y,X),q} X^{(k)}_{j-q} + \sum_{q=1}^{n} m_{(Y,X'),q} X'^{(k)}_{j-q} &= \sum_{q=1}^{n} m_{(Y,X),q} X^{(k)}_{j-q} + \sum_{q=1}^{n} m_{(Y,X),1-q} X'^{(k)}_{j-q} \\ &= \sum_{q=1}^{n} m_{(Y,X),q} X^{(k)}_{j-q} + \sum_{q=1}^{n} m_{(Y,X),n+1-q} X'^{(k)}_{j-q} \\ &= \sum_{q=1}^{n} m_{(Y,X),q} X^{(k)}_{j-q} + \sum_{q=1}^{n} m_{(Y,X),q} X'^{(k)}_{j-n-1+q} \\ &= \sum_{q=1}^{n} m_{(Y,X),q} \left(X^{(k)}_{j-q} + X'^{(k)}_{j-1+q} \right) \\ &= \sum_{q=1}^{n} m_{(Y,X),q} \left(X^{(k)}_{j-q} + X'^{(k)}_{j-q} - X'^{(k)}_{j-q} - X^{(k)}_{j-q+1} \right. \\ &\quad \left. + X^{(k)}_{j-q+1} + \ldots + X^{(k)}_{j-1+q} + X'^{(k)}_{j-1+q} \right) \\ &= \sum_{q=1}^{n} m_{(Y,X),q} \left(2 \sum_{l=1-q}^{q} I^{(k)}_{j-l} - 2 \sum_{l=1-q}^{q-1} H^{(k)}_{j-l} \right) \end{aligned}$$

We can write the components for the I and the H separately now.

$$\begin{aligned}
\sum_{q=1}^{n} 2m_{(Y,X),q}\left(\sum_{l=1-q}^{q} I_{j-l}^{(k)}\right) &= \sum_{q=1}^{n} 2m_{(Y,X),q}\left(\sum_{l=1-q}^{0} I_{j-l}^{(k)} + \sum_{l=1}^{q} I_{j-l}^{(k)}\right)\\
&= \sum_{q=1}^{n} 2m_{(Y,X),q}\left(\sum_{l=n+1-q}^{n} I_{j-l}^{(k)} + \sum_{l=1}^{q} I_{j-l}^{(k)}\right)\\
&= 2m_{(Y,X),1}\left(I_{j-n}^{(k)} + I_{j-1}^{(k)}\right) + 2m_{(Y,X),2}\left(I_{j-n+1}^{(k)} + I_{j-n}^{(k)} + I_{j-1}^{(k)} + I_{j-2}^{(k)}\right)\\
&\quad + \ldots + 2m_{(Y,X),n}\left(I_{j-1}^{(k)} + \ldots I_{j-n}^{(k)} + I_{j-1}^{(k)} + \ldots I_{j-n}^{(k)}\right)\\
&= \sum_{q=1}^{n} 2\left(\sum_{l=q}^{n} m_{(Y,X),l} + \sum_{l=n-q+1}^{n} m_{(Y,X),l}\right) I_{j-q}^{(k)} .
\end{aligned}$$

Similarly,

$$\begin{aligned}
\sum_{q=1}^{n} -2m_{(Y,X),q}\left(\sum_{l=1-q}^{q-1} H_{j-l}^{(k)}\right) &= \sum_{q=1}^{n} -2m_{(Y,X),q}\left(\sum_{l=n+1-q}^{n} H_{j-l}^{(k)} + \sum_{l=1}^{q-1} H_{j-l}^{(k)}\right)\\
&= \sum_{q=1}^{n} -2\left(\sum_{l=q+1}^{n} m_{(Y,X),l} + \sum_{l=n-q+1}^{n} m_{(Y,X),l}\right) H_{j-q}^{(k)} .
\end{aligned}$$

If Y is a new vertex, coming from old vertices E and E' and which is H-like, then with the components given in lemma 4.2 and the formal notation given in the text of this lemma,

$$Y_j^{(k+1)} = m_{Y,1}A^{(k)} + \sum_{Z\notin\{X,X'\}} \sum_{q=1}^{n} m_{(Y,Z),q} Z_{j-q}^{(k)} + \sum_{q=1}^{n} m_{(Y,X),q} X_{j-q}^{(k)} + \sum_{q=1}^{n} m_{(Y,X'),q} X'^{(k)}_{j-q} .$$

Remind the formal notation are

$$\begin{aligned}
m_{(Y,X),q} &= (m_{(E,X),q} + m_{(E',X),q-1})/2 ,\\
m_{(Y,X'),q} &= (m_{(E,X'),q} + m_{(E',X'),q-1})/2 .
\end{aligned}$$

From lemma 2.6, both E and X belonging to the floor,

$$\begin{aligned}
m_{(Y,X),q} &= (m_{(E',X'),-q} + m_{(E,X'),1-q})/2\\
&= m_{(Y,X'),1-q} .
\end{aligned}$$

We find the same relationship with Y being an old vertex belonging to the basement. As a consequence, the same proof may be run with the formal notation $m_{(Y,X),q}$ and $m_{(Y,X'),q}$.

Now, let Y be an old vertex which belongs to the ceiling. As we have written in the proof of lemma 2.4,

$$Y_j^{(k+1)} = m_{Y,1}A^{(k)} + \sum_{Z\notin\{X,X'\}} \sum_{q=1}^{n} m_{(Y,Z),q} Z_{j-q}^{(k)} + \sum_{q=1}^{n} m_{(Y,X),q} X_{j-q}^{(k)} + \sum_{q=1}^{n} m_{(Y,X'),q} X'^{(k)}_{j-q} .$$

And, from lemma 2.6,

$$m_{(Y,X),q} = m_{(Y,X'),-q}$$

So,

$$\begin{aligned}
\sum_{q=1}^{n} m_{(Y,X),q} X_{j-q}^{(k)} + \sum_{q=1}^{n} m_{(Y,X'),q} X'^{(k)}_{j-q} &= \sum_{q=1}^{n} m_{(Y,X),q} X_{j-q}^{(k)} + \sum_{q=1}^{n} m_{(Y,X),-q} X'^{(k)}_{j-q} \\
&= \sum_{q=1}^{n} m_{(Y,X),q} \left(X_{j-q}^{(k)} + X'^{(k)}_{j+q} \right) \\
&= \sum_{q=1}^{n} m_{(Y,X),q} \left(X_{j-q}^{(k)} + X'^{(k)}_{j-q} - X'^{(k)}_{j-q} - X_{j-q+1}^{(k)} \right. \\
&\quad \left. + X_{j-q+1}^{(k)} + \ldots + X_{j+q}^{(k)} + X'^{(k)}_{j+q} \right) \\
&= \sum_{q=1}^{n} m_{(Y,X),q} \left(2 \sum_{l=-q}^{q} I_{j-l}^{(k)} - 2 \sum_{l=-q}^{q-1} H_{j-l}^{(k)} \right)
\end{aligned}$$

We can write the components for the I and the H separately now.

$$\begin{aligned}
\sum_{q=1}^{n} 2m_{(Y,X),q} \left(\sum_{l=-q}^{q} I_{j-l}^{(k)} \right) &= \sum_{q=1}^{n} 2m_{(Y,X),q} \left(\sum_{l=-q}^{0} I_{j-l}^{(k)} + \sum_{l=1}^{q} I_{j-l}^{(k)} \right) \\
&= \sum_{q=1}^{n} 2m_{(Y,X),q} \left(\sum_{l=n-q}^{n} I_{j-l}^{(k)} + \sum_{l=1}^{q} I_{j-l}^{(k)} \right) \\
&= 2m_{(Y,X),1} \left(I_{j-n+1}^{(k)} + I_{j-n}^{(k)} + I_{j-1}^{(k)} \right) \\
&\quad + 2m_{(Y,X),2} \left(I_{j-n+2}^{(k)} + I_{j-n+1}^{(k)} + I_{j-n}^{(k)} + I_{j-1}^{(k)} + I_{j-2}^{(k)} \right) \\
&\quad + \ldots + 2m_{(Y,X),n} \left(I_{j-n}^{(k)} + I_{j-1}^{(k)} \ldots I_{j-n}^{(k)} + I_{j-1}^{(k)} + \ldots I_{j-n}^{(k)} \right) \\
&= \sum_{q=1}^{n} 2 \left(\sum_{l=n-q}^{n} m_{(Y,X),l} + \sum_{l=q}^{n} m_{(Y,X),l} \right) I_{j-q}^{(k)} .
\end{aligned}$$

Similarly,

$$\begin{aligned}
\sum_{q=1}^{n} -2m_{(Y,X),q} \left(\sum_{l=-q}^{q-1} H_{j-l}^{(k)} \right) &= \sum_{q=1}^{n} -2m_{(Y,X),q} \left(\sum_{l=n-q}^{n} H_{j-l}^{(k)} + \sum_{l=1}^{q-1} H_{j-l}^{(k)} \right) \\
&= \sum_{q=1}^{n} -2 \left(\sum_{l=q+1}^{n} m_{(Y,X),l} + \sum_{l=n-q}^{n} m_{(Y,X),l} \right) H_{j-q}^{(k)} .
\end{aligned}$$

If Y is a new vertex, coming from old vertices E and E' and which is I-like then, with the components given in lemma 4.2 and the formal notation given in the text of this lemma,

$$Y_j^{(k+1)} = m_{Y,1} A^{(k)} + \sum_{Z \notin \{X,X'\}} \sum_{q=1}^{n} m_{(Y,Z),q} Z_{j-q}^{(k)} + \sum_{q=1}^{n} m_{(Y,X),q} X_{j-q}^{(k)} + \sum_{q=1}^{n} m_{(Y,X'),q} X'^{(k)}_{j-q} .$$

Remind the formal notation are

$$\begin{aligned}
m_{(Y,X),q} &= (m_{(E,X),q} + m_{(E',X),q})/2 , \\
m_{(Y,X'),q} &= (m_{(E,X'),q} + m_{(E',X'),q})/2 .
\end{aligned}$$

From lemma 2.6, both E and X belonging to the floor,

$$\begin{aligned} m_{(Y,X),q} &= (m_{(E',X'),-q} + m_{(E,X'),-q})/2 \\ &= m_{(Y,X'),-q} \ . \end{aligned}$$

We find the same relationship with Y being an old vertex belonging to the ceiling. As a consequence, the same proof may be run with the formal notation $m_{(Y,X),q}$ and $m_{(Y,X'),q}$. ∎

Remark The components $m_{(Y,H),q}$ and $m_{(Y,I),q}$ given in this lemma are not the only possible choice. Indeed, the sets of new vertices I and J are linked by the following equality:

$$\sum_{q=1}^{n} I_{j-q} - H_{j-q} = 0 \ .$$

Here, for example, to go from $X^{(k)}_{j-q}$ to $X'^{(k)}_{j-1+q}$ we turn in the same way regardless of the relative position of these two vertices, even if we have to cross almost all the indices. Thus, another possible choice is for all Y being an old vertex belonging to the basement or a new vertex being H-like, and for all $q \in \{1 \dots n\}$,

$$\begin{aligned} m_{(Y,H),q} &:= -2\left(\sum_{l=q+1}^{n} m_{(Y,X),l} + \sum_{l=n-q+1}^{n} m_{(Y,X),l}\right) \\ &\quad +2\sum_{l=1}^{n} m_{(Y,X),l} \\ &= 2\left(\sum_{l=1}^{q} m_{(Y,X),l} - \sum_{l=n-q+1}^{n} m_{(Y,X),l}\right) \\ m_{(Y,I),q} &:= 2\left(\sum_{l=q}^{n} m_{(Y,X),l} + \sum_{l=n-q+1}^{n} m_{(Y,X),l}\right) \\ &\quad -2\sum_{l=1}^{n} m_{(Y,X),l} \\ &= 2\left(\sum_{l=q}^{n} m_{(Y,X),l} - \sum_{l=1}^{n-q} m_{(Y,X),l}\right) \ , \end{aligned}$$

and for all Y being an old vertex belonging to the ceiling, or a new vertex which is I-like, and for all $q \in \{1 \dots n\}$,

$$\begin{aligned} m_{(Y,H),q} &:= -2\left(\sum_{l=q+1}^{n} m_{(Y,X),l} + \sum_{l=n-q}^{n} m_{(Y,X),l}\right) \\ &\quad +2\sum_{l=1}^{n} m_{(Y,X),l} \\ &= 2\left(\sum_{l=1}^{q} m_{(Y,X),l} - \sum_{l=n-q}^{n} m_{(Y,X),l}\right) \end{aligned}$$

$$
\begin{aligned}
m_{(Y,I),q} &:= 2\left(\sum_{l=q}^{n} m_{(Y,X),l} + \sum_{l=n-q}^{n} m_{(Y,X),l}\right) \\
&\quad -2\sum_{l=1}^{n} m_{(Y,X),l} \\
&= 2\left(\sum_{l=q}^{n} m_{(Y,X),l} - \sum_{l=1}^{n-q-1} m_{(Y,X),l}\right) .
\end{aligned}
$$

In practice, this choice cannot be taken blindly. For instance in the Catmull-Clark scheme, we have for all $q \in \{1, \ldots, n-1\}$ $m_{(D,E),q} = 0$. So, if $q < n$,

$$
\begin{aligned}
m_{(D,H),q} &= -2\left(\sum_{l=q+1}^{n} m_{(D,X),l} + \sum_{l=n-q+1}^{n} m_{(D,X),l}\right) \\
&= -2\left(m_{(D,E),n} + m_{(D,E),n}\right) \\
&= -4m_{(D,E),n} ,
\end{aligned}
$$

and if $q = n$

$$
\begin{aligned}
m_{(D,H),q} &= -2\left(\sum_{l=q+1}^{n} m_{(D,X),l} + \sum_{l=n-q+1}^{n} m_{(D,X),l}\right) \\
&= -2\left(0 + m_{(D,E),n}\right) \\
&= -2m_{(D,E),n} .
\end{aligned}
$$

But we have also

$$
\begin{aligned}
m_{(D,I),q} &= 2\left(\sum_{l=q}^{n} m_{(Y,X),l} + \sum_{l=n-q+1}^{n} m_{(Y,X),l}\right) \\
&= 2\left(m_{(D,E),n} + m_{(D,E),n}\right) \\
&= 4m_{(D,E),n} ,
\end{aligned}
$$

whatever q is. That means that

$$
\begin{aligned}
D_j^{(k+1)} &= \ldots + \sum_{q=1}^{n-1} -4m_{(D,E),n} H_{j-q}^{(k)} + (-2m_{(D,E),n}) H_{j-n}^{(k)} + \sum_{q=1}^{n} 4m_{(D,E),n} I_{j-q}^{(k)} \\
&= \ldots + 4m_{(D,E),n} \sum_{q=1}^{n} \left(I_{j-q}^{(k)} - H_{j-q}^{(k)}\right) - \left(-2m_{(D,E),n} H_j^{(k)}\right) \\
&= \ldots + 2m_{(D,E),n} H_j^{(k)} .
\end{aligned}
$$

So, in practice, it would be of great advantage to choose for all $q \in \{1, \ldots, n-1\}$

$$m_{(D,H),q} = 0$$

and

$$m_{(D,H),n} = 2m_{(D,E),n}$$

and for all $q \in \{1, \ldots, n\}$

$$m_{(D,I),q} = 0$$

Lemma 4.4 *If the scheme is rotationally and mirror invariant, if for all X belonging to the floor, we substitute in the vector $\mathbf{P}^{(k)}$ the vertices X_j and X'_j by*

$$H_j^{(k)} = \frac{X_j^{(k)} + X_{j-1}'^{(k)}}{2} \qquad \text{and} \qquad I_j^{(k)} = \frac{X_j^{(k)} + X_j'^{(k)}}{2} \ ,$$

and, in addition, if we choose the difference of phases given in lemma 4.1 for the old vertices which belong to the basement or the ceiling, as well as for the new vertices which are H-like or I-like, then all the components of the new subdivision matrices in the frequency domain $\tilde{\mathrm{M}}(\omega)$ are real.

Proof First of all, the components of the new subdivision matrix in the spatial domain M are defined as real linear combinations of the components of the old subdivision matrix. Thus, from lemma 2.4 we can deduce that the the new subdivision matrix in the frequency domain $\tilde{\mathrm{M}}(0)$ is real.

As a consequence, for proving this lemma, we just have to check that the new subdivision matrix in the spatial domain M is rotationally and mirror invariant. Consequently, lemma 4.1 may be applied leading to the result.

By construction, the new subdivision matrix M is rotationally invariant: we have defined the new components $M_{l_{(X,j)},l_{(Y,h)}}$ as $m_{(X,Y),j-h}$. So we just have to check that its components follow the relationship given in lemma 2.6.

If X and Y belong to the basement or to the ceiling, then $m_{(X,Y),q}$ have not changed and follow the relationship given in lemma 2.6.

Let X be H-like coming from E and E'. From lemma 4.2 we get

$$m_{(X,Y),q} \;=\; (m_{(E,Y),q} + m_{(E',Y),q-1})/2 \ .$$

If Y an old vertex belonging to the basement, from lemma 2.4,

$$\begin{aligned} m_{(X,Y),q} &= (m_{(E',Y),-1-q} + m_{(E,Y),-q})/2 \\ &= m_{(X,Y),-q} \ . \end{aligned}$$

If Y an old vertex belonging to the ceiling, from lemma 2.4,

$$\begin{aligned} m_{(X,Y),q} &= (m_{(E',Y),-q} + m_{(E,Y),1-q})/2 \\ &= m_{(X,Y),1-q} \ . \end{aligned}$$

If Y a new H-like vertex coming from F F', from lemma 4.3,

$$\begin{aligned} m_{(X,Y),q} &= -2\left(\sum_{l=q+1}^{n} m_{(X,F),l} + \sum_{l=n-q+1}^{n} m_{(X,F),l}\right) \\ &= m_{(X,Y),n-q} \ . \end{aligned}$$

If Y a new I-like vertex coming from F F', from lemma 4.3,

$$\begin{aligned} m_{(X,Y),q} &= 2\left(\sum_{l=q}^{n} m_{(X,F),l} + \sum_{l=n-q+1}^{n} m_{(X,F),l}\right) \\ &= m_{(X,Y),n+1-q} \ . \end{aligned}$$

Now, let X be I-like coming from E and E'. From lemma 4.2 we get

$$m_{(X,Y),q} = (m_{(E,Y),q} + m_{(E',Y),q})/2 .$$

If Y is an old vertex belonging to the basement, from lemma 2.4,

$$\begin{aligned} m_{(X,Y),q} &= (m_{(E',Y),-1-q} + m_{(E,Y),-1-q})/2 \\ &= m_{(X,Y),-1-q} . \end{aligned}$$

If Y is an old vertex belonging to the ceiling, from lemma 2.4,

$$\begin{aligned} m_{(X,Y),q} &= (m_{(E',Y),-q} + m_{(E,Y),-q})/2 \\ &= m_{(X,Y),-q} . \end{aligned}$$

If Y is a new H-like vertex coming from F F', from lemma 4.3,

$$\begin{aligned} m_{(X,Y),q} &= -2\left(\sum_{l=q+1}^{n} m_{(X,F),l} + \sum_{l=n-q}^{n} m_{(X,F),l}\right) \\ &= m_{(X,Y),n-1-q} . \end{aligned}$$

If Y is a new I-like vertex coming from F F', from lemma 4.3,

$$\begin{aligned} m_{(X,Y),q} &= 2\left(\sum_{l=q}^{n} m_{(X,F),l} + \sum_{l=n-q}^{n} m_{(X,F),l}\right) \\ &= m_{(X,Y),n-q} . \end{aligned}$$

As a consequence, the new subdivision matrix in the spatial domain M is rotationally and mirror invariant if the new H-like vertices are considered as belonging to the basement and the new I-like vertices to the ceiling. So, if the differences of phases $\phi_X - \phi_Y$ are chosen like in lemma 4.1, the matrices in the frequency domain $\tilde{\mathrm{M}}(\omega)$ are real. ∎

Remark Zorin and Schröder proposed another way to get real subdivision matrices in the frequency domain [18]. Basically, they put all the vertices on the ceiling, even the vertices which belong to the basement (actually they stay in the frequency domain and propose new frequency components as linear combination of existing ones). In a way this is more systematic but it is much more expensive if we need to know the eigenvectors. Indeed, their work concerned only the eigenvalues, so this was not a problem for them.

4.3 Necessary Conditions on the Real Matrices

Of course, if we process the eigenanalysis on the new real matrices, we cannot directly apply the conditions given in Sect. 3 on their eigenelements.

More precisely, the eigenanalysis of the new real matrices in the frequency domain provide the same eigenvalues, but eigenvector components

$$(\tilde{v}_q(\omega))_H \quad \text{and} \quad (\tilde{v}_q(\omega))_I$$

instead of

$$(\tilde{v}_q(\omega))_X \quad \text{and} \quad (\tilde{v}_q(\omega))_{X'} .$$

In the following lemma, we define formal components $(\tilde{v}_q(\omega))_X$ and $(\tilde{v}_q(\omega))_{X'}$ from $(\tilde{v}_q(\omega))_H$ and $(\tilde{v}_q(\omega))_I$ given by the eigenanalysis of the new real matrices. These formal components are defined such that if we exchange the components $(\tilde{v}_q(\omega))_H$ and $(\tilde{v}_q(\omega))_I$ of the eigenvectors with them, the conditions given in Sect. 3 applied on these new eigenvectors are necessary conditions for the C^2-convergence of the scheme.

Lemma 4.5 *Let $\tilde{\mathbf{v}}_1(0)$ be the eigenvector associated with the main eigenvalue of the new real matrix in the frequency domain $\tilde{\mathrm{M}}(0)$ built in Sect.4.2. If for every H-like and I-like new vertex coming from E and E', we exchange the components $(\tilde{v}_1(0))_H$ and $(\tilde{v}_1(0))_I$ by*

$$(\tilde{v}_1(0))_E := (\tilde{v}_1(0))_I = (\tilde{v}_1(0))_H$$

and

$$(\tilde{v}_1(0))_{E'} := (\tilde{v}_1(0))_E$$

then the conditions given in Sect. 3.1 applied on this new eigenvector are necessary conditions for the C^0-convergence of the scheme.

Proof Let H and I be new vertices from E and E'. From lemma 2.7 noting that $\rho_E^{(k)} = \rho_{E'}^{(k)}$,

$$\lim_{k\to\infty}(I_j^{(k)}) = \frac{1}{2}\left(\lim_{k\to\infty}(E_j^{(k)}) + \lim_{k\to\infty}(E_j'^{(k)})\right) = \mathcal{F},$$

and so

$$\lim_{k\to\infty}(\tilde{I}^{(k)}(\omega)) = n\mathcal{F}\delta_{\omega,0}.$$

Similarly,

$$\lim_{k\to\infty}(H_j^{(k)}) = \frac{1}{2}\left(\lim_{k\to\infty}(E_j^{(k)}) + \lim_{k\to\infty}(E_{j-1}'^{(k)})\right) = \mathcal{F},$$

and so

$$\lim_{k\to\infty}(\tilde{H}^{(k)}(\omega)) = n\mathcal{F}\delta_{\omega,0}.$$

From lemma 2.3, we deduce that

$$\mathcal{F} = \frac{1}{n}\sum_{q\in L_1(0)} \mathcal{P}(q,0)\,(\tilde{v}_q(0))_I\,.$$

Assuming that $L_1(0) = \{1\}$,

$$\mathcal{F} = \frac{1}{n}\mathcal{P}(1,0)\,(\tilde{v}_1(0))_I\,.$$

And the classical condition on $(\tilde{v}_1(0))_E$ should be

$$\mathcal{F} = \frac{1}{n}\mathcal{P}(1,0)\,(\tilde{v}_1(0))_E\,.$$

So, we just have to define $(\tilde{v}_1(0))_E$ as

$$(\tilde{v}_1(0))_E := (\tilde{v}_1(0))_I$$

Similarly, we could define it as

$$(\tilde{v}_1(0))_E := (\tilde{v}_1(0))_H$$

∎

Lemma 4.6 *Because we do not in practice define $\phi_{E'}$ and ϕ_E, we can assume that $\phi_{E'} = \alpha_{E'}$ and $\phi_E = \alpha_E$.*

Let $\tilde{\mathbf{v}}_1(\pm 1)$ be the eigenvector associated with the main eigenvalue of the new real matrices in the frequency domain $\tilde{\mathrm{M}}(\pm 1)$ built in Sect.4.2. If for every H-like and I-like new vertex coming from E and E', we exchange the components $(\tilde{v}_1(\pm 1))_H$ and $(\tilde{v}_1(\pm 1))_I$ by

$$
\begin{aligned}
(\tilde{v}_1(\pm 1))_E &:= 2\frac{(\tilde{v}_1(\pm 1))_H}{\exp\left(\frac{2i\pi(\phi_E-\phi_H)}{n}\right)+\exp\left(\frac{2i\pi(\phi_{E'}-\phi_H-1)}{n}\right)} \\
&:= 2\frac{(\tilde{v}_1(\pm 1))_I}{\exp\left(\frac{2i\pi(\phi_E-\phi_I)}{n}\right)+\exp\left(\frac{2i\pi(\phi_{E'}-\phi_I)}{n}\right)}
\end{aligned}
$$

and

$$(\tilde{v}_1(0))_{E'} := (\tilde{v}_1(0))_E$$

then the conditions given in Sect. 3.2 applied on this new eigenvector are necessary conditions for the C^1-convergence of the scheme.

Proof Let H and I be new vertices form E and E'. From lemma 2.8 noting that $\rho_E^{(k)} = \rho_{E'}^{(k)}$, $\forall j \in \{1,\ldots,n\}$,

$$
\begin{aligned}
&\lim_{k\to\infty}\left(\left(\frac{I_j^{(k)}-\mathcal{F}^{(k)}(0,0)}{\varrho_E^{(k)}}\right)-\right.\\
&\left.\left(\frac{\cos(\theta_{(E,j,k)})+\cos(\theta_{(E',j,k)})}{2}\frac{\partial\mathcal{F}}{\partial x}(0,0)+\frac{\sin(\theta_{(E,j,k)})+\sin(\theta_{(E',j,k)})}{2}\frac{\partial\mathcal{F}}{\partial y}(0,0)\right)\right)\\
&=\lim_{k\to\infty}\left(\frac{1}{2}\left(\frac{E_j^{(k)}-\mathcal{F}^{(k)}}{\rho_E^{(k)}}+\frac{E_j'^{(k)}-\mathcal{F}^{(k)}}{\rho_{E'}^{(k)}}\right)-\right.\\
&\left.\left(\frac{\cos(\theta_{(E,j,k)})+\cos(\theta_{(E',j,k)})}{2}\frac{\partial\mathcal{F}}{\partial x}(0,0)+\frac{\sin(\theta_{(E,j,k)})+\sin(\theta_{(E',j,k)})}{2}\frac{\partial\mathcal{F}}{\partial y}(0,0)\right)\right)\\
&=0\,.
\end{aligned}
$$

So, from lemmas 2.1 and 2.2, and because $\theta_{(E,j,k)} = \frac{2\pi}{n}(j+\alpha_E+\alpha_k)$

$$
\begin{aligned}
&\lim_{k\to\infty}\left(\frac{\tilde{I}^{(k)}(\omega)-n\mathcal{F}^{(k)}\,\delta_{\omega,0}}{\rho_E^{(k)}}\right.\\
&-n\frac{\mathcal{F}_x-\mathrm{i}\mathcal{F}_y}{2}\frac{\exp\left(\frac{2\mathrm{i}\pi}{n}(\alpha_E+\alpha_k-\phi_I-\phi_k)\right)+\exp\left(\frac{2\mathrm{i}\pi}{n}(\alpha_{E'}+\alpha_k-\phi_I-\phi_k)\right)}{2}\delta_{\omega,1}\\
&\left.-\,n\frac{\mathcal{F}_x+\mathrm{i}\mathcal{F}_y}{2}\frac{\exp\left(\frac{2i\pi}{n}(\alpha_E+\alpha_k-\phi_I-\phi_k)\right)+\exp\left(\frac{2i\pi}{n}(\alpha_{E'}+\alpha_k-\phi_I-\phi_k)\right)}{2}\delta_{\omega,-1}\right)=0\,.
\end{aligned}
$$

From lemma 2.3, we deduce that

$$
\begin{aligned}
&\lim_{k\to\infty}\left(n\frac{\mathcal{F}_x-i\mathcal{F}_y}{2}\frac{\exp\left(\frac{2i\pi}{n}(\alpha_E+\alpha_k-\phi_I-\phi_k)\right)+\exp\left(\frac{2i\pi}{n}(\alpha_{E'}+\alpha_k-\phi_I-\phi_k)\right)}{2}\right.\\
&\left.-\frac{\tilde{\lambda}_1(\pm 1)^k}{\rho_E^{(k)}}\sum_{q\in L_1(\pm 1)}\mathcal{P}(q,\pm 1)\,(\tilde{v}_q(\pm 1))_I\right)=0
\end{aligned}
$$

Assuming that $L_1(\pm 1) = \{1\}$,

$$\lim_{k\to\infty}\left(n\frac{\mathcal{F}_x \mp \mathrm{i}\mathcal{F}_y}{2}\frac{\exp\left(\frac{2\mathrm{i}\pi}{n}(\alpha_E+\alpha_k-\phi_I-\phi_k)\right)+\exp\left(\frac{2\mathrm{i}\pi}{n}(\alpha_{E'}+\alpha_k-\phi_I-\phi_k)\right)}{2} - \frac{\tilde{\lambda}_1(\pm 1)^k}{\rho_E^{(k)}}\mathcal{P}(1,\pm 1)\left(\tilde{v}_1(\pm 1)\right)_I\right) = 0$$

And the classical condition on $\left(\tilde{v}_1(\pm 1)\right)_E$ should be

$$\lim_{k\to\infty}\left(n\frac{\mathcal{F}_x \mp \mathrm{i}\mathcal{F}_y}{2}\exp\left(\frac{2\mathrm{i}\pi}{n}\left(\alpha_E+\alpha_k-\phi_I-\phi_k\right)\right) - \frac{\tilde{\lambda}_1(\pm 1)^k}{\rho_E^{(k)}}\mathcal{P}(1,\pm 1)\left(\tilde{v}_1(\pm 1)\right)_E\right) = 0$$

So, we just have to define formally $\left(\tilde{v}_1(\pm 1)\right)_E$ such that

$$\begin{aligned}&n\frac{\mathcal{F}_x \mp \mathrm{i}\mathcal{F}_y}{2}\frac{\exp\left(\frac{2\mathrm{i}\pi}{n}(\alpha_E+\alpha_k-\phi_I-\phi_k)\right)+\exp\left(\frac{2\mathrm{i}\pi}{n}(\alpha_{E'}+\alpha_k-\phi_I-\phi_k)\right)}{2}\\ &= n\frac{\mathcal{F}_x \mp \mathrm{i}\mathcal{F}_y}{2}\frac{\left(\tilde{v}_1(\pm 1)\right)_I}{\left(\tilde{v}_1(\pm 1)\right)_E}\exp\left(\frac{2\mathrm{i}\pi}{n}\left(\alpha_E+\alpha_k-\phi_I-\phi_k\right)\right)\end{aligned}$$

that is

$$\begin{aligned}\left(\tilde{v}_1(\pm 1)\right)_E &:= \frac{2\exp\left(\frac{2\mathrm{i}\pi}{n}\left(\alpha_E-\phi_I\right)\right)}{\exp\left(\frac{2\mathrm{i}\pi}{n}(\alpha_E-\phi_I)\right)+\exp\left(\frac{2\mathrm{i}\pi}{n}(\alpha_{E'}-\phi_I)\right)}\left(\tilde{v}_1(\pm 1)\right)_I\\ &= \frac{2}{\exp\left(\frac{2\mathrm{i}\pi}{n}(\phi_E-\phi_I)\right)+\exp\left(\frac{2\mathrm{i}\pi}{n}(\alpha_{E'}-\phi_I+\phi_E-\alpha_E)\right)}\left(\tilde{v}_1(\pm 1)\right)_I\end{aligned}$$

Because we do not in practice define $\phi_{E'}$ and ϕ_E, we can assume that $\phi_{E'} = \alpha_{E'}$ and $\phi_E = \alpha_E$ and so we define formally $\left(\tilde{v}_1(\pm 1)\right)_E$ as

$$\left(\tilde{v}_1(\pm 1)\right)_E := \frac{2}{\exp\left(\frac{2\mathrm{i}\pi}{n}(\phi_E-\phi_I)\right)+\exp\left(\frac{2\mathrm{i}\pi}{n}(\phi_{E'}-\phi_I)\right)}\left(\tilde{v}_1(\pm 1)\right)_I$$

Besides, from lemma 2.7 noting that $\rho_E^{(k)} = \rho_{E'}^{(k)}$, $\forall j \in \{1,\ldots,n\}$,

$$\begin{aligned}&\lim_{k\to\infty}\left(\left(\frac{H_j^{(k)}-\mathcal{F}^{(k)}(0,0)}{\varrho_E^{(k)}}\right) -\right.\\ &\left.\left(\frac{\cos(\theta_{(E,j,k)})+\cos(\theta_{(E',j-1,k)})}{2}\frac{\partial\mathcal{F}}{\partial x}(0,0)+\frac{\sin(\theta_{(E,j,k)})+\cos(\theta_{(E',j-1,k)})}{2}\frac{\partial\mathcal{F}}{\partial y}(0,0)\right)\right)\\ &= \lim_{k\to\infty}\left(\frac{1}{2}\left(\frac{E_j^{(k)}-\mathcal{F}^{(k)}}{\rho_E^{(k)}}+\frac{E_{j-1}'^{(k)}-\mathcal{F}^{(k)}}{\rho_{E'}^{(k)}}\right)\right.\\ &\left.-\left(\frac{\cos(\theta_{(E,j,k)})+\cos(\theta_{(E',j-1,k)})}{2}\frac{\partial\mathcal{F}}{\partial x}(0,0)+\frac{\sin(\theta_{(E,j,k)})+\cos(\theta_{(E',j-1,k)})}{2}\frac{\partial\mathcal{F}}{\partial y}(0,0)\right)\right)\\ &= 0\,.\end{aligned}$$

So, from lemma 2.1,

$$\lim_{k\to\infty}\left(\frac{\tilde{H}^{(k)}(\omega)-n\mathcal{F}^{(k)}\,\delta_{\omega,0}}{\rho_E^{(k)}}-\right.$$
$$n\frac{\mathcal{F}_x-\mathrm{i}\mathcal{F}_y}{2}\,\frac{\exp\left(\frac{2\mathrm{i}\pi}{n}(\alpha_E+\alpha_k-\phi_H-\phi_k)\right)+\exp\left(\frac{2\mathrm{i}\pi}{n}(\alpha_{E'}+\alpha_k-1-\phi_H-\phi_k)\right)}{2}\,\delta_{\omega,1}$$
$$\left.-\,n\frac{\mathcal{F}_x+\mathrm{i}\mathcal{F}_y}{2}\,\frac{\exp\left(\frac{2i\pi}{n}(\alpha_E+\alpha_k-\phi_H-\phi_k\right)+\exp\left(\frac{2i\pi}{n}(\alpha_{E'}+\alpha_k-1-\phi_H-\phi_k\right)}{2}\,\delta_{\omega,-1}\right)=0\ .$$

So, we can similarly define $(\tilde{v}_1(\pm 1))_E$ as

$$(\tilde{v}_1(\pm 1))_E := \frac{2}{\exp\left(\frac{2\mathrm{i}\pi}{n}(\phi_E-\phi_H)\right)+\exp\left(\frac{2\mathrm{i}\pi}{n}(\phi_{E'}-\phi_H-1)\right)}\,(\tilde{v}_1(\pm 1))_H$$

∎

Remark For making the new frequency matrices real, we have chosen

$$\phi_H = 0 \quad \text{and} \quad \phi_I = 1/2\ .$$

Furthermore if we choose

$$\phi_{E'} = 1-\phi_E = \phi$$

then, the formal definition of $(\tilde{v}_1(\pm 1))_E$ becomes

$$\begin{aligned}(\tilde{v}_1(1))_E &:= \frac{(\tilde{v}_1(1))_H}{\cos(\frac{2\pi\phi}{n})}\\ &:= \frac{(\tilde{v}_1(1))_I}{\cos(\frac{2\pi(\phi-1/2)}{n})}\ .\end{aligned}$$

Furthermore,

$$\begin{aligned}\cos(\frac{2\pi(\phi-1/2)}{n}) &= \cos(\frac{2\pi\phi}{n}-\frac{\pi}{n})\\ &= \cos(\frac{2\pi\phi}{n})\cos(\frac{\pi}{n})+\sin(\frac{2\pi\phi}{n})\sin(\frac{\pi}{n})\end{aligned}$$

So,

$$\frac{(\tilde{v}_1(1))_I}{(\tilde{v}_1(1))_E}=\frac{(\tilde{v}_1(1))_H}{(\tilde{v}_1(1))_E}\cos(\frac{\pi}{n})+\sqrt{1-\left(\frac{(\tilde{v}_1(1))_H}{(\tilde{v}_1(1))_E}\right)^2}\sin(\frac{\pi}{n})$$
$$\frac{1}{\sin(\frac{\pi}{n})^2}\left(\frac{(\tilde{v}_1(1))_I-(\tilde{v}_1(1))_H\cos(\frac{\pi}{n})}{(\tilde{v}_1(1))_E}\right)^2=1-\left(\frac{(\tilde{v}_1(1))_H}{(\tilde{v}_1(1))_E}\right)^2$$

which leads to

$$(\tilde{v}_1(1))_E=(\tilde{v}_1(1))_{E'}=\sqrt{{(\tilde{v}_1(1))_H}^2+\left(\frac{(\tilde{v}_1(1))_I-(\tilde{v}_1(1))_H\cos(\pi/4)}{\sin(\pi/4)}\right)^2}$$

Lemma 4.7 *Because we do not in practice define $\phi_{E'}$ and ϕ_E, we can assume that $\phi_{E'} = \alpha_{E'}$ and $\phi_E = \alpha_E$.*

Let $\tilde{\mathbf{v}}_2(0)$ be the eigenvector associated with the sub-dominant eigenvalue of the new real matrix in the frequency domain $\tilde{\mathrm{M}}(0)$ built in Sect.4.2, and $\tilde{\mathbf{v}}_1(\pm 2)$ the eigenvectors associated with the main eigenvalue of the new real matrices in the frequency domain $\tilde{\mathrm{M}}(\pm 2)$ built in Sect.4.2. If for every H-like and I-like new vertex coming from E and E', we exchange on the one hand the components $(\tilde{v}_2(0))_H$ and $(\tilde{v}_2(0))_I$ by

$$(\tilde{v}_2(0))_E \quad := \quad (\tilde{v}_2(0))_I = (\tilde{v}_2(0))_H$$

and

$$(\tilde{v}_2(0))_{E'} := (\tilde{v}_2(0))_E$$

and, on the other hand, $(\tilde{v}_1(\pm 2))_H$ and $(\tilde{v}_1(\pm 2))_I$ by

$$\begin{aligned}(\tilde{v}_1(\pm 2))_E \quad &:= \quad 2\frac{(\tilde{v}_1(\pm 2))_H}{\exp\left(2\frac{2i\pi(\phi_E-\phi_H)}{n}\right)+\exp\left(2\frac{2i\pi(\phi_{E'}-\phi_H-1)}{n}\right)} \\ &:= \quad 2\frac{(\tilde{v}_1(\pm 2))_I}{\exp\left(2\frac{2i\pi(\phi_E-\phi_I)}{n}\right)+\exp\left(2\frac{2i\pi(\phi_{E'}-\phi_I)}{n}\right)}\end{aligned}$$

and

$$(\tilde{v}_1(\pm 2))_{E'} := (\tilde{v}_1(\pm 2))_E$$

then the conditions given in Sect. 3.3 applied on this new eigenvector are necessary conditions for the C^2-convergence of the scheme.

Proof Let H and I be new vertices form E and E'. From lemma 2.9 noting that $\rho_E^{(k)} = \rho_{E'}^{(k)}$, $\forall j \in \{1,\ldots,n\}$,

$$\begin{aligned}&\lim_{k\to\infty}\left(\frac{I_j^{(k)}-\mathcal{F}^{(k)}(0,0)-\varrho_E^{(k)}\left(\cos(\theta_{(E,j,k)})\frac{\partial\mathcal{F}^{(k)}}{\partial x}(0,0)-\sin(\theta_{(E,j,k)})\frac{\partial\mathcal{F}^{(k)}}{\partial y}(0,0)\right)}{{\varrho_E^{(k)}}^2}\right.\\ &\left[\left(\frac{\partial^2\mathcal{F}}{\partial x^2}(0,0)+\frac{\partial^2\mathcal{F}}{\partial y^2}(0,0)\right)\frac{1}{4}+\frac{\partial^2\mathcal{F}}{\partial x\partial y}(0,0)\frac{\sin(2\theta_{(E,j,k)})+\sin(2\theta_{(E',j,k)})}{2}\right.\\ &\left.\left.+\left(\frac{\partial^2\mathcal{F}}{\partial x^2}(0,0)-\frac{\partial^2\mathcal{F}}{\partial y^2}(0,0)\right)\frac{\cos(2\theta_{(E,j,k)})+\cos(2\theta_{(E',j,k)})}{2}\right]\right)\\ &=\lim_{k\to\infty}\left(\frac{1}{2}\frac{\Delta_{E,j}^{(k)}+\Delta_{E',j}^{(k)}}{{\varrho_E^{(k)}}^2}\right.\\ &\left[\left(\frac{\partial^2\mathcal{F}}{\partial x^2}(0,0)+\frac{\partial^2\mathcal{F}}{\partial y^2}(0,0)\right)\frac{1}{4}+\frac{\partial^2\mathcal{F}}{\partial x\partial y}(0,0)\frac{\sin(2\theta_{(E,j,k)})+\sin(2\theta_{(E',j,k)})}{2}\right.\\ &\left.\left.+\left(\frac{\partial^2\mathcal{F}}{\partial x^2}(0,0)-\frac{\partial^2\mathcal{F}}{\partial y^2}(0,0)\right)\frac{\cos(2\theta_{(E,j,k)})+\cos(2\theta_{(E',j,k)})}{2}\right]\right)\\ &=0\,.\end{aligned}$$

with

$$\Delta_{X,j}^{(k)} := X_j^{(k)}-\mathcal{F}^{(k)}(0,0)-\varrho_X^{(k)}\left(\cos(\theta_{(X,j,k)})\frac{\partial\mathcal{F}^{(k)}}{\partial x}(0,0)-\sin(\theta_{(X,j,k)})\frac{\partial\mathcal{F}^{(k)}}{\partial y}(0,0)\right)\,.$$

So, from lemma 2.1,

$$\lim_{k\to\infty}\left(\frac{1}{{\rho_E^{(k)}}^2}\left(\tilde{I}^{(k)}(\omega)-n\mathcal{F}^{(k)}\,\delta_{\omega,0}\right.\right.$$
$$-\rho_E^{(k)}n\frac{\mathcal{F}_x^{(k)}-\mathrm{i}\mathcal{F}_y^{(k)}}{2}\frac{\exp\left(\frac{2\mathrm{i}\pi}{n}(\alpha_E+\alpha_k-\phi_I-\phi_k\right)+\exp\left(\frac{2\mathrm{i}\pi}{n}(\alpha_{E'}+\alpha_k-\phi_I-\phi_k\right)}{2}\,\delta_{\omega,1}$$
$$\left.-\rho_E^{(k)}n\frac{\mathcal{F}_x^{(k)}+i\mathcal{F}_y^{(k)}}{2}\frac{\exp\left(-\frac{2\mathrm{i}\pi}{n}(\alpha_E+\alpha_k-\phi_I-\phi_k\right)+\exp\left(-\frac{2\mathrm{i}\pi}{n}(\alpha_{E'}+\alpha_k-\phi_I-\phi_k\right)}{2}\,\delta_{\omega,-1}\right)$$
$$-n\frac{\mathcal{F}_{xx}+\mathcal{F}_{yy}}{4}\,\delta_{\omega,0}$$
$$+\frac{\frac{\mathcal{F}_{xx}-\mathcal{F}_{yy}}{4}-\mathrm{i}\frac{\mathcal{F}_{xy}}{2}}{2}\frac{\exp\left(2\frac{2\mathrm{i}\pi}{n}(\alpha_E+\alpha_k-\phi_I-\phi_k\right)+\exp\left(2\frac{2\mathrm{i}\pi}{n}(\alpha_{E'}+\alpha_k-\phi_I-\phi_k\right)}{2}\,\delta_{\omega,2}$$
$$\left.+\frac{\frac{\mathcal{F}_{xx}-\mathcal{F}_{yy}}{4}+\mathrm{i}\frac{\mathcal{F}_{xy}}{2}}{2}\frac{\exp\left(-2\frac{2\mathrm{i}\pi}{n}(\alpha_E+\alpha_k-\phi_I-\phi_k\right)+\exp\left(-2\frac{2\mathrm{i}\pi}{n}(\alpha_{E'}+\alpha_k-\phi_I-\phi_k\right)}{2}\,\delta_{\omega,-2}\right)=0\ .$$

From lemma 2.3, we deduce that assuming that $L_2(0)=\{2\}$,

$$\mathcal{F}_{xx}+\mathcal{F}_{yy}=\frac{4}{n}{\nu_{\pm1}}^2\mathcal{P}(2,0)\frac{[(\tilde{v}_2(0))_I-(\tilde{v}_2(0))_A]}{|(\tilde{v}_1(1))_E|^2}$$

And the classical condition on $(\tilde{v}_2(0))_E$ should be

$$\mathcal{F}_{xx}+\mathcal{F}_{yy}=\frac{4}{n}{\nu_{\pm1}}^2\mathcal{P}(2,0)\frac{[(\tilde{v}_2(0))_E-(\tilde{v}_2(0))_A]}{|(\tilde{v}_1(1))_E|^2}$$

So, we just have to define $(\tilde{v}_2(0))_E$ as

$$(\tilde{v}_2(0))_E:=(\tilde{v}_2(0))_I$$

Similarly, we could define it as

$$(\tilde{v}_2(0))_E:=(\tilde{v}_2(0))_H$$

From lemma 2.3, we also get that, assuming $L_1(2)=\{1\}$,

$$\lim_{k\to\infty}\left((\mathcal{F}_{xx}-\mathcal{F}_{yy}\mp\mathrm{i}2\mathcal{F}_{xy})\frac{\exp^{2\frac{2i\pi}{n}(\alpha_E+\alpha_k-\phi_I-\phi_k)}+\exp^{2\frac{2i\pi}{n}(\alpha_{E'}+\alpha_k-\phi_I-\phi_k)}}{2}\right.$$
$$\left.-\frac{8}{n}{\nu_{\pm1}}^2\mathcal{P}(1,\pm2)\frac{(\tilde{v}_1(\pm2))_I}{|(\tilde{v}_1(\pm1))_E|^2}\right)=0$$

And the classical condition on $(\tilde{v}_1(2))_E$ should be

$$\lim_{k\to\infty}\left((\mathcal{F}_{xx}-\mathcal{F}_{yy}\mp\mathrm{i}2\mathcal{F}_{xy})\exp\left(\frac{4\mathrm{i}\pi}{n}(\alpha_E+\alpha_k-\phi_E-\phi_k)\right)-\frac{8}{n}{\nu_{\pm1}}^2\mathcal{P}(1,\pm2)\frac{(\tilde{v}_1(\pm2))_E}{|(\tilde{v}_1(\pm1))_E|^2}\right)=0\ .$$

So, because we assume that $\phi_E = \alpha_E$ and $\phi_{E'} = \alpha_{E'}$, we just have to define $\left(\tilde{v}_1(2)\right)_E$ as

$$\left(\tilde{v}_1(2)\right)_E := \frac{2}{\exp^{2\frac{2i\pi}{n}(\phi_E - \phi_I)} + \exp^{2\frac{2i\pi}{n}(\phi_{E'} - \phi_I)}} \left(\tilde{v}_1(2)\right)_I$$

Similarly, we could define it as

$$\left(\tilde{v}_1(2)\right)_E := \frac{2}{\exp^{2\frac{2i\pi}{n}(\phi_E - \phi_H)} + \exp^{2\frac{2i\pi}{n}(\phi_{E'} - \phi_H - 1)}} \left(\tilde{v}_1(2)\right)_H$$

$\blacksquare$

Remark As for the previous remark, if we choose

$$\phi_H = 0 \quad \text{and} \quad \phi_I = 1/2\ .$$

and

$$\phi_{E'} = 1 - \phi_E$$

then, the formal definition of $\left(\tilde{v}_1(2)\right)_E$ becomes

$$\left(\tilde{v}_1(2)\right)_E = \frac{\left(\tilde{v}_1(2)\right)_H}{\cos(2\frac{2\pi\phi_E}{n})}$$

$$\left(\tilde{v}_1(2)\right)_E = \frac{\left(\tilde{v}_1(2)\right)_H}{2\cos(\frac{2\pi\phi_E}{n})^2 - 1}$$

thus, with lemma 4.6

$$\left(\tilde{v}_1(2)\right)_E = \frac{\left(\tilde{v}_1(2)\right)_H}{2\left(\frac{\left(\tilde{v}_1(1)\right)_H}{\left(\tilde{v}_1(1)\right)_E}\right)^2 - 1}$$

4.4 Sanity Check

In this section we will check these new necessary conditions. We consider the Catmull-Clark scheme [4] around an ordinary mark point A. Firstly, we write the transformed subdivision matrices with the original vicinity of A. We verify that this matrix cannot be real. Then we write the transformed subdivision matrices with the new vicinity of A where every vertex lies on a symmetry axis. We verify that, with appropriate phases, these matrices are real. Finally, we check that the Catmull-Clark scheme around an ordinary mark point satisfies the new necessary conditions.

The Original Transformed Subdivision Matrices For the phases associated with each set of vertices, we choose:

$$\begin{aligned} \phi_B &= \phi_D = 0 \\ \phi_C &= \phi_F = \frac{1}{2} \\ \phi_{E'} &= 1 - \phi_E = 1 - \phi \end{aligned}$$

Then, with

$$\text{if } \omega \neq 0, \quad \tilde{\mathbf{P}}^{(k)}(\omega) = \begin{bmatrix} \tilde{B}^{(k)}(\omega) \\ \tilde{C}^{(k)}(\omega) \\ \tilde{D}^{(k)}(\omega) \\ \tilde{E}^{(k)}(\omega) \\ \tilde{E'}^{(k)}(\omega) \\ \tilde{F}^{(k)}(\omega) \end{bmatrix} \quad \text{and otherwise} \quad \tilde{\mathbf{P}}^{(k)}(0) = \begin{bmatrix} \tilde{A}^{(k)}(0) \\ \tilde{B}^{(k)}(0) \\ \tilde{C}^{(k)}(0) \\ \tilde{D}^{(k)}(0) \\ \tilde{E}^{(k)}(0) \\ \tilde{E'}^{(k)}(0) \\ \tilde{F}^{(k)}(0) \end{bmatrix}$$

we get

$$\tilde{\mathrm{M}}(0) = \frac{1}{64} \begin{bmatrix} 36 & 24 & 4 & 0 & 0 & 0 & 0 \\ 24 & 32 & 8 & 0 & 0 & 0 & 0 \\ 16 & 32 & 16 & 0 & 0 & 0 & 0 \\ 6 & 38 & 12 & 6 & 1 & 1 & 0 \\ 4 & 28 & 24 & 4 & 4 & 0 & 0 \\ 4 & 28 & 24 & 4 & 0 & 4 & 0 \\ 1 & 12 & 36 & 2 & 6 & 6 & 1 \end{bmatrix}$$

and

$$\tilde{\mathrm{M}}(\omega) = \frac{1}{64} \begin{bmatrix} 24 + 8\cos(\pi\omega/2) & 8\cos(\pi\omega/4) & 0 & 0 & 0 & 0 \\ 32\cos(\pi\omega/4) & 16 & 0 & 0 & 0 & 0 \\ 36 + 2\cos(\pi\omega/2) & 12\cos(\pi\omega/4) & 6 & \exp^{i\pi\omega\phi/2} & \exp^{-i\pi\omega\phi/2} & 0 \\ 24\exp^{-\frac{i\pi\omega}{2}\phi} + 4\exp^{\frac{i\pi\omega}{2}(1-\phi)} & 24\exp^{\frac{i\pi\omega}{2}(\frac{1}{2}-\phi)} & 4\exp^{-\frac{i\pi\omega}{2}\phi} & 4 & 0 & 0 \\ 24\exp^{\frac{i\pi\omega}{2}\phi} + 4\exp^{-\frac{i\pi\omega}{2}(1-\phi)} & 24\exp^{-\frac{i\pi\omega}{2}(\frac{1}{2}-\phi)} & 4\exp^{\frac{i\pi\omega}{2}\phi} & 0 & 4 & 0 \\ 12\cos(\pi\omega/4) & 36 & 2\cos(\pi\omega/4) & 6\exp^{-\frac{i\pi\omega}{2}(\frac{1}{2}-\phi)} & 6\exp^{\frac{i\pi\omega}{2}(\frac{1}{2}-\phi)} & 1 \end{bmatrix}$$

No ϕ lets the matrices $\tilde{\mathrm{M}}(\omega)$ be real.

The New Transformed Subdivision Matrices We first give the new real matrices and then their eigenelements. With

$$\text{if } \omega \neq 0, \quad \tilde{\mathbf{P}}^{(k)}(\omega) = \begin{bmatrix} \tilde{B}^{(k)}(\omega) \\ \tilde{C}^{(k)}(\omega) \\ \tilde{D}^{(k)}(\omega) \\ \tilde{H}^{(k)}(\omega) \\ \tilde{I}^{(k)}(\omega) \\ \tilde{F}^{(k)}(\omega) \end{bmatrix} \quad \text{and otherwise} \quad \tilde{\mathbf{P}}^{(k)}(0) = \begin{bmatrix} \tilde{A}^{(k)}(0) \\ \tilde{B}^{(k)}(0) \\ \tilde{C}^{(k)}(0) \\ \tilde{D}^{(k)}(0) \\ \tilde{H}^{(k)}(0) \\ \tilde{I}^{(k)}(0) \\ \tilde{F}^{(k)}(0) \end{bmatrix}$$

and

$$\begin{aligned} \phi_H &= 0 \\ \phi_I &= \frac{1}{2} \end{aligned}$$

we get

$$\tilde{\mathrm{M}}(0) = \frac{1}{64} \begin{bmatrix} 36 & 24 & 4 & 0 & 0 & 0 & 0 \\ 24 & 32 & 8 & 0 & 0 & 0 & 0 \\ 16 & 32 & 16 & 0 & 0 & 0 & 0 \\ 6 & 38 & 12 & 6 & 2 & 0 & 0 \\ 4 & 28 & 24 & 4 & 4 & 0 & 0 \\ 4 & 28 & 24 & 4 & 0 & 4 & 0 \\ 1 & 12 & 36 & 2 & 0 & 12 & 1 \end{bmatrix}$$

and

$$\tilde{\mathrm{M}}(\omega) = \frac{1}{64} \begin{bmatrix} 24 + 8\cos(\pi\omega/2) & 8\cos(\pi\omega/4) & 0 & 0 & 0 & 0 \\ 32\cos(\pi\omega/4) & 16 & 0 & 0 & 0 & 0 \\ 36 + 2\cos(\pi\omega/2) & 12\cos(\pi\omega/4) & 6 & 2 & 0 & 0 \\ 24 + 4\cos(\pi\omega/2) & 24\cos(\pi\omega/4) & 4 & 4 & 0 & 0 \\ 28\cos(\pi\omega/4) & 24 & 4\cos(\pi\omega/4) & 0 & 4 & 0 \\ 12\cos(\pi\omega/4) & 36 & 2\cos(\pi\omega/4) & 0 & 12 & 1 \end{bmatrix}$$

which are real.

The eigenvalues are

$$\tilde{\lambda}_1(0) = 1 \quad \tilde{\lambda}_2(0) = \frac{1}{4} \quad \tilde{\lambda}_1(1) = \frac{1}{2} \quad \tilde{\lambda}_1(2) = \frac{1}{4}$$

The eigenvectors are

$$\tilde{v}_1(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \tilde{v}_2(0) = \begin{bmatrix} -2 \\ 1 \\ 4 \\ 10 \\ 13 \\ 13 \\ 22 \end{bmatrix} \quad \tilde{v}_1(1) = \begin{bmatrix} 1 \\ \sqrt{2} \\ 2 \\ 2 \\ 2.1213 \\ 2.8284 \end{bmatrix} \quad \tilde{v}_1(2) = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 3 \\ 4 \\ 8 \end{bmatrix}$$

The New Conditions applied on the New Eigenvectors The eigenvalues satisfy to the necessary conditions given in Sect. 3.

From lemma 4.6, with $\phi_H = 0$, $\phi_I = 1/2$ and $\phi_{E'} = 1 - \phi_E$, we get

$$|(\tilde{v}_1(1))_E| = |(\tilde{v}_1(1))_{E'}| = \sqrt{|(\tilde{v}_1(1))_H|^2 + \left(\frac{|(\tilde{v}_1(1))_I| - |(\tilde{v}_1(1))_H|\cos(\pi/4)}{\sin(\pi/4)}\right)^2}$$

so,

$$\begin{aligned} |(\tilde{v}_1(1))_E| &= |(\tilde{v}_1(1))_{E'}| \\ &= \sqrt{4 + \left(\frac{2.1213 - 2\cos(\pi/4)}{\sin(\pi/4)}\right)^2} \\ &= \sqrt{5} \end{aligned}$$

From lemma 4.7, we get

$$(\tilde{v}_2(0))_E = (\tilde{v}_2(0))_I = (\tilde{v}_2(0))_H$$

so,

$$(\tilde{v}_2(0))_E = 13$$

From lemma 4.7, with $\phi_H = 0$ and $\phi_{E'} = 1 - \phi_E$, we get

$$(\tilde{v}_1(2))_E = \frac{(\tilde{v}_1(2))_H}{2\left(\frac{(\tilde{v}_1(1))_H}{(\tilde{v}_1(1))_E}\right)^2 - 1}$$

so,

$$\begin{aligned}(\tilde{v}_1(2))_E &= \frac{3}{2\left(\frac{2}{\sqrt{5}}\right)^2 - 1} \\ &= 5\end{aligned}$$

Then, the following vectors

$$\tilde{v}_1(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \tilde{v}_2(0) = \begin{bmatrix} -2 \\ 1 \\ 4 \\ 10 \\ 13 \\ 13 \\ 22 \end{bmatrix} \quad \tilde{v}_1(1) = \begin{bmatrix} 1 \\ \sqrt{2} \\ 2 \\ \sqrt{5} \\ \sqrt{5} \\ 2.8284 \end{bmatrix} \quad \tilde{v}_1(2) = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 5 \\ 5 \\ 8 \end{bmatrix}$$

should satisfy to the new necessary conditions. They do. Indeed,

- $\forall X \in \{A, B, C, D, E, E', F\}$,

$$(\tilde{v}_1(0))_X = \nu_0 = 1$$

- $\forall X \in \{B, C, D, E, E', F\}$,

  $|(\tilde{v}_1(1))_X| = |(\tilde{v}_1(-1))_X|$ and they have to be sorted as the parameters ϱ_X

  And more precisely, having $\tilde{\lambda}_1(1) = \tilde{\lambda}_1(-1) = 1/2$, and fixing $\nu_1 = \nu_{-1} = 1$,

$$\varrho_X^{(k)} = \frac{|(\tilde{v}_1(1))_X|}{2^k} = \frac{|(\tilde{v}_1(-1))_X|}{2^k} .$$

  The map between the mesh and this parameter space is injective, as expected.

- $\forall X \in \{B, C, D, E, E', F\}$,

$$\frac{(\tilde{v}_2(0))_X - (\tilde{v}_2(0))_A}{(\tilde{v}_1(1))_X^2} = \frac{(\tilde{v}_2(0))_X - (\tilde{v}_2(0))_A}{(\tilde{v}_1(-1))_X^2} = \frac{\nu_{20}}{\nu_1^2} = \nu_{20} = 3 .$$

- $\forall X \in \{B, C, D, E, E', F\}$,

$$\frac{(\tilde{v}_1(2))_X}{(\tilde{v}_1(1))_X^2} = \frac{\nu_{+21}}{\nu_1^2} = \nu_{+21} = 1 , \qquad \frac{(\tilde{v}_1(2))_X}{(\tilde{v}_1(-1))_X^2} = \frac{\nu_{+21}}{\nu_{-1}^2} = \nu_{+21} = 1 ,$$

$$\frac{(\tilde{v}_1(-2))_X}{(\tilde{v}_1(1))_X^2} = \frac{\nu_{-21}}{\nu_1^2} = \nu_{-21} = 1 , \qquad \frac{(\tilde{v}_1(-2))_X}{(\tilde{v}_1(-1))_X^2} = \frac{\nu_{-21}}{\nu_{-1}^2} = \nu_{-21} = 1 .$$

5 Conclusion

In this paper we have presented practical conditions for tuning a scheme in order to control its artifacts in the vicinity of a mark point. To do so, we have looked for good behaviour of the limit vertices rather than good mathematical properties of the limit surface. The good behaviour of the limit vertices is characterised by the definition of C^2-convergence of a scheme. Because this definition is theoretic and formal, we have proposed necessary explicit conditions. These conditions are applied at any mark point being a vertex of valency n or the centre of a n-sided face with n greater or equal to three.

The necessary conditions for C^2-convergence of a scheme that we have proposed in this paper concern the eigenvalues and eigenvectors of subdivision matrices in the frequency domain. The components of these matrices may be complex. Having them real would simplify numerical analysis of the eigenstructure of the matrices, especially in the context of scheme tuning where we manipulate symbolic terms. In this paper we have shown that an appropriate choice of the parameter space combined with a substitution of vertices lets us transform these matrices into pure real ones. The substitution consists in replacing some vertices by linear combinations of themselves. But the conditions given above cannot be applied directly on the new pure real matrices. So we have explained how to derive conditions on the eigenelements of the real matrices which are necessary for the C^2-convergence of the scheme.

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