Recent developments in LCF: examples of structural induction

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Recent Developments in LCF: Examples of Structural Induction

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Abstract

Manna and Waldinger have outlined a large proof that probably exceeds the power of current automatic theorem-provers. The proof establishes the unification algorithm for terms composed of variables, constants, and other terms. Two theorems from this proof, involving structural induction, are performed in the LCF proof assistant. These theorems concern a function that searches for an occurrence of one term inside another, and a function that lists the variables in a term.

Formally, terms are regarded as abstract syntax trees. LCF automatically builds the first-order theory, with equality, of this recursive data structure.

The first theorem has a simple proof: induction followed by rewriting. The second theorem requires a cases split and substitution throughout the goal. Each theorem is proved by reducing the initial goal to simpler and simpler subgoals. LCF provides many standard proof strategies for attacking goals; the user can program additional ones in LCF's meta-language, ML. This flexibility allows the user to take ideas from such diverse fields as denotational semantics and logic programming.
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1. Introduction

An interactive proof assistant should be able to reason about a variety of data structures and algorithms, and automatically perform simple proofs. It should be flexible, allowing experimentation with different ways of expressing and performing proofs. These are the goals of the proof assistant Edinburgh LCF (Gordon, Milner, Wadsworth [1979]).

In view of the success of the Boyer and Moore [1979] theorem-prover, why should a proof assistant excite any interest? Manna and Waldinger [1981, page 47] manually prove the Unification Algorithm and remark, "Although the above proof may be beyond the power of current automatic systems, a partially interactive system could be used to produce it with known techniques. This approach requires more human effort, but it still would convey many of the benefits of automatic synthesis." This paper presents two theorems from Manna and Waldinger's theory.

2. Essential Background

LCF relies on one fundamental principle: proofs are conducted in a meta-language, ML. ML is a general-purpose functional program-
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A language whose data values include terms, formulas, and
theorems of the logic. Inference rules are ML functions that map
theorems to theorems. The only way an ML program can prove a
theorem is by applying inference rules to axioms or previously
proved theorems.

LCF's logic, PPLAMBDA, uses Scott's theory of continuous partial
orders (Stoy [1977]). PPLAMBDA has the usual introduction and
elimination rules for each connective. Please note its ASCII
representation of logical formulas (Figure 1).

---

**PPLAMBDA Terms**
- `c`: constant, where c is a constant symbol
- `x`: variable
- `\x.t`: lambda-abstraction over a term
- `t u`: combination (application of function to argument)
- `UU`: "bottom" or "undefined" element
- `TT`: truth-value "true"
- `FF`: truth-value "false"

**PPLAMBDA Formulas**
- `t == u`: equality of t and u
- `!x.A`: universal quantifier
- `?x.A`: existential quantifier
- `A \& B`: conjunction
- `A \lor B`: disjunction
- `A => B`: implication
- `A <=> B`: if-and-only-if
- `~A`: negation

Figure 1. Syntax of the logic PPLAMBDA
3. Axiomatising a Structure: Combinator Terms

Let us formalise Manna and Waldinger's [1981] theory of the Unification Algorithm, which concerns substitution over combinator terms. These are like PPLAMBDA terms without lambda-abstraction. For proofs we are only concerned with their abstract syntax:

\[ \text{term} = \text{CONST const} \mid \text{VAR var} \mid \text{COMB term term} \]

To axiomatise this structure in LCF, we introduce the abstract types "const", "var", and "term", then axiomatise them using LCF's structure package. LCF stores the types and axioms on a theory file, which can become part of a theory hierarchy.

% percent signs enclose comments%

new_type 0 'var';; %declare the types var, const, term%
new_type 0 'const';;
new_type 0 'term';;
struct_axm (":term", %build the theory of terms%
  'strict',
  ['CONST', ['c:const'];
   'VAR', ['v:var'];
   'COMB', ['t1:term'; 't2:term']});

The resulting theory includes the constructor functions CONST, VAR, and COMB, and axioms stating that these are distinct, one-to-one, etc. The constructor functions are all strict; for instance, CONST UU == UU. To build a theory that includes infinite and partially defined structures, call struct_axm with argument 'lazy' instead of 'strict'.

4. The Occurrence Relation

Our proofs concern the ordering relation "t OCCS u", an infix function that tests whether \( t \) occurs in \( u \) as a sub-structure. This requires a theory of the infix equality function, \( = \). The structure package can prove theorems describing the outcome of the equality test\(^1\) for various arguments; for instance, if \( v, t, u, t', u' \) are all defined, then

\[
\begin{align*}
(\text{COMB } t \text{ u}) = (\text{COMB } t' \text{ u'}) & \iff (t = t') \text{ AND } (u = u') \\
(\text{VAR } v) = (\text{COMB } t \text{ u}) & \equiv \text{ FF}
\end{align*}
\]

Figure 2 shows how to axiomatise the OCCS and OCCS_EQ functions, binding the axioms to ML names. The function OCCS_EQ tests "equals or occurs in." The function OCCS is defined as a set of clauses, one for each possible input:\(^2\) a term cannot occur in a constant or variable, and occurs in COMB \( t_1 \text{ t}_2 \) exactly if it equals or occurs in \( t_1 \) or \( t_2 \). This style of defining functions, reminiscent of Prolog (Clocksin and Mellish [1981]) or HOPE (Burstall et al. [1981]), eliminates the need for destructor and discriminator functions.

\(^1\) Do not confuse the function \( = \) with the predicate \( \equiv \). The formula "\( x = y \)" which may be proved using inference rules, asserts that \( x \) and \( y \) are equal. The term "\( x \equiv y \)" represents a computable equality test applied to \( x \) and \( y \). Likewise, do not confuse the truth-valued functions AND, OR, and NOT, with the logical connectives \( \land \), \( \lor \), and \( \neg \).

\(^2\) The first clause asserts that OCCS is strict; the other uses of \( \text{UU} \) confine the clauses to defined values only. Our theorems contain similar definedness hypotheses, a reflection that they were originally formulated for a first-order logic, not for PPLAMBDA.
let OCCS_EQ =
  new_axiom (`OCCS_EQ`,
`!t t2. t OCCS_EQ t2 == (t=t2) OR (t OCCS t2)`);

let OCCS_CLAUSES =
  new_axiom (`OCCS_CLAUSES`,
`!t. t OCCS UU == UU
  /
  (!c. ~ c==UU ==> t OCCS (CONST c) == FF)
  /
  (!v. ~ v==UU ==> t OCCS (VAR v) == FF)
  /
  (!t1 t2. ~ t1==UU ==> ~ t2==UU ==> t OCCS (COMB t1 t2) == (t OCCS_EQ t1) OR (t OCCS_EQ t2))`);

Figure 2. Axioms for the infix functions OCCS_EQ and OCCS.

5. The Variables Proposition

Our first theorem concerns a function VARS_OF, which computes a list of all the variables in a term. (We use a theory of lists, with constructors NIL and CONS, and infix operators APP for append, MEM for membership test.)

let VARS_OF_CLAUSES =
  new_axiom (`VARS_OF_CLAUSES`,
`VAR_OF UU == UU
  /
  (!c. ~ c==UU ==> VARS_OF(CONST c) == NIL)
  /
  (!v. ~ v==UU ==> VARS_OF(VAR v) == CONS v NIL)
  /
  (!t u. ~ t==UU ==> ~ u==UU ==> VARS_OF(COMB t u) == (VARS_OF t) APP (VARS_OF u))`);

Let us prove that a variable v occurs in a term t exactly when v is a member of the list VARS_OF(t). We give LCF the goal:
set_goal ([],
    "!v. v=UU =>
    !t. v MEM (VARS_OF t) = (VAR v) OCCS_EQ t"
);;

We will work backwards from the goal, by applying subgoaling functions, called tactics, to it. A tactic returns a list of subgoals, paired with a proof function that maps proofs of the subgoals to a proof of the original goal. By applying further tactics we reduce all the subgoals to trivial ones. Then we assemble the complete proof from the proof functions.

One simple tactic is GEN_TAC, which reasons that to prove !x.A(x) it suffices to choose a new variable x' and prove A(x'), since this theorem can then be generalised over x'. Another tactic is DISCH_TAC, which reasons that to prove A=>B, it suffices to prove B under the assumption A, since this assumption can then be discharged. Notation: the double bar means "suffices to prove"; assumptions are enclosed in [square brackets]; other assumptions of the goal are implicitly passed to the subgoals.

\[
\begin{align*}
!x. A(x) & \quad \text{GEN_TAC} \quad \text{GEN_TAC} \\
\equiv & \quad \equiv \\
A(x') & \quad \text{GEN_TAC}
\end{align*}
\]

\[
\begin{align*}
A=>B & \quad \text{DISCH_TAC} \\
\equiv & \quad \text{DISCH_TAC} \\
[A] B & \quad \text{DISCH_TAC}
\end{align*}
\]

For interactive proof, LCF's subgoal package is convenient: it stacks pending and solved subgoals, displays the current goal, and applies the proof functions in the correct order. You can apply tactics and back up from faulty steps.
5.1. The structural induction tactic

The structure package provides a tactic, TERM_INDUCT_TAC, to perform structural induction on a goal \( \textit{lt} \cdot A(t) \). This produces four subgoals: \( t \) may be a COMB (the step case); \( t \) may be a VAR or CONST (the base cases); \( t \) may be UU (the undefined case). The subgoals include induction hypotheses and assumptions that the sub-structures are defined.

\[
\text{\texttt{lt} \cdot A(t) ~ \text{TERM_INDUCT_TAC}}
\]
\[
\begin{align*}
\text{\texttt{[ A(t1) ; A(t2) ; \sim t1==UU ; \sim t2==UU ]}} & \quad \text{A(COMB t1 t2)} \\
\text{\texttt{[ \sim v==UU ]}} & \quad \text{A(VAR v)} \\
\text{\texttt{[ \sim c==UU ]}} & \quad \text{A(CONST c)} \\
\text{\texttt{A(UU)}} &
\end{align*}
\]

Let us ask the subgoal package to \texttt{expand} the current goal using TERM_INDUCT_TAC. The induction variable \( t \) is submerged inside the goal, so the tactic calls GEN_TAC and DISCH_TAC before applying induction.

\[
\text{expand (TERM_INDUCT_TAC "t");;}
\]

5.2. The rewriting tactic

LCF prints the four resulting subgoals and their assumptions (Figure 3). We can prove each one by \texttt{rewriting}: if we have a theorem \( t==u \), change the goal by replacing every instance \( t' \) of \( t \) by the corresponding instance \( u' \). For \texttt{implicative} rewrites, a theorem \( A=>(t==u) \) may be used to rewrite an instance \( t' \) by \( u' \) if the antecedent \( A' \) can be proved.
"v MEM (VARS_OF (COMB t1 t2)) == (VAR v) OCXS_EQ (COMB t1 t2)"
[ "v == uu" ]
[ "v MEM (VARS_OF t1) == (VAR v) OCXS_EQ t1" ]
[ "v MEM (VARS_OF t2) == (VAR v) OCXS_EQ t2" ]
[ "t1 == uu" ]
[ "t2 == uu" ]

"v MEM (VARS_OF (VAR v')) == (VAR v) OCXS_EQ (VAR v')"
[ "v == uu" ]
[ "v' == uu" ]

"v MEM (VARS_OF (CONST c)) == (VAR v) OCXS_EQ (CONST c)"
[ "v == uu" ]
[ "c == uu" ]

"v MEM (VARS_OF uu) == (VAR v) OCXS_EQ uu"
[ "v == uu" ]

Figure 3. Subgoals after applying induction.

The most interesting case involves terms of the form
(COMB t1 t2), with induction hypotheses for t1 and t2. The left
and right sides converge:

v MEM (VARS_OF (COMB t1 t2))
   unfolding the definition of VARS_OF --->
v MEM ((VARS_OF t1) APP (VARS_OF t2))
   by a theorem about MEM, APP, and OR --->
   (v MEM (VARS_OF t1)) OR (v MEM (VARS_OF t2))

(VAR v) OCXS_EQ (COMB t1 t2)
   unfolding the definition of OCXS_EQ --->
   (VAR v)=(COMB t1 t2) OR
   (((VAR v) OCXS_EQ t1) OR ((VAR v) OCXS_EQ t2))
   since any VAR is distinct from any COMB --->
   (((VAR v) OCXS_EQ t1) OR ((VAR v) OCXS_EQ t2))
   by the induction hypotheses --->
   (v MEM (VARS_OF t1)). OR (v MEM (VARS_OF t2))

The other goals converge similarly. To perform such reasoning,
LFC provides the tactic ASM_REWRITE_TAG. This rewrites the goal
using its assumptions and a list of theorems furnished by the
user. The symbols AND_CLAUSES, OR_CLAUSES, etc., denote axioms
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and theorems from parent theories.

```lisp
expand (ASM_REWRITE_TAC
[AND_CLAUSES; OR_CLAUSES;
TERM_EQUAL_ALL;
MEM_CLAUSES; MEM_SINGLE; MEM_APP;
VARS_OF_CLAUSES; VARS_OF_TOTAL;
OCCS_EQ_CLAUSES])
```

LCF reports that the subgoal is solved and prints those that remain. The above call of ASM_REWRITE_TAC includes enough theorems to solve any of the four subgoals.

5.3. Summarising the proof

Now that the interactive proof is complete, let us combine the tactics we used into a composite one that performs the entire proof. For combining tactics, LCF provides functions called tactics. The basic ones are THEN, ORElse, and REPEAT.

- **TAC1 THEN TAC2**
  calls TAC1, then applies TAC2 to all resulting subgoals
  
- **TAC1 ORElse TAC2**
  calls TAC1, if it fails then calls TAC2

- **REPEAT TAC**
  calls TAC repeatedly on the goal and its subgoals

The tactic that proves the Variables theorem is

```lisp
TERM_INDUCT_TAC "t" THEN
ASM_REWRITE_TAC [AND_CLAUSES; OR_CLAUSES; etc.]
```

In words, the proof is induction followed by rewriting. Many proofs have this simple form — for instance, properties of list
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utilities (append, map, membership), and totality of recursively
defined functions.

Given a set of theorems, ASM_REWRITE_TAC strips off universal
quantifiers, splits apart conjunctions, and decides which of the
resulting pieces are useful for rewriting or for solving implica-
tive rewrites. It accepts not only term rewrites, \( t = u \), but also
formula rewrites, \( A \leftrightarrow B \). After rewriting, it removes tautologies
from the goal -- perhaps solving it completely, returning an
empty subgoal list.

Edinburgh LCF provided a similar tactic, SIMPTAC, consisting of
seven inscrutable pages of ML. Since SIMPTAC was impossible to
modify, Avra Cohn [1982] spent considerable effort adapting her
proofs to its limitations. In contrast, ASM_REWRITE_TAC has a
modular construction. Its apparently baroque strategy is con-
trolled by a twelve-line ML function that calls tactics, pattern
matchers, canonical form translators, rewriting functions, and
tautology checkers. These components can easily be changed to
suit individual needs or correct shortcomings.

6. Transitivity of the Occurrence Relation

Let us prove a more difficult theorem, that the ordering relation
OCCS is transitive:

\[
\begin{align*}
!ta. \sim ta =u U & \Rightarrow \\
!tb. ta \text{ OCCS } tb = TT & \Rightarrow \\
!tc. tb \text{ OCCS } tc = TT & \Rightarrow ta \text{ OCCS } tc = TT
\end{align*}
\]
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If we induct on the variable tc, rewriting solves only three of its four subgoals. The COMB case remains:

\[
\frac{
((tb = t1) \text{ OR } (tb \text{ OCCS } t1)) \text{ OR } 
((tb = t2) \text{ OR } (tb \text{ OCCS } t2)) \quad \Rightarrow 
((ta = t1) \text{ OR } (ta \text{ OCCS } t1)) \text{ OR } 
((ta = t2) \text{ OR } (ta \text{ OCCS } t2)) \quad \text{ == } TT
}{
[ "\text{ ~ } ta \text{ == } UU" ]
[ "ta \text{ OCCS } tb \text{ == } TT" ]
[ "tb \text{ OCCS } t1 \text{ == } TT \Rightarrow ta \text{ OCCS } t1 \text{ == } TT" ]
[ "tb \text{ OCCS } t2 \text{ == } TT \Rightarrow ta \text{ OCCS } t2 \text{ == } TT" ]
[ "\text{ ~ } t1 \text{ == } UU" ]
[ "\text{ ~ } t2 \text{ == } UU" ]
}\]

The goal has the form \( A \Rightarrow B \), so we can use DISCH_TAC to attempt proving \( B \) by assuming the antecedent \( A \). Since \( B \) has the form \( b1 \text{ OR } b2 \text{ OR } b3 \text{ OR } b4 \text{ == } TT \), it suffices to prove that one of \( b1 \), \( b2 \), \( b3 \), or \( b4 \) equals \( TT \). The antecedent \( A \) has the form \( a1 \text{ OR } a2 \text{ OR } a3 \text{ OR } a4 \text{ == } TT \), and, by studying the assumptions, we realise that if any \( ai \) equals \( TT \), then the corresponding \( bi \) must equal \( TT \).

If \( A \) were a disjunction \( A1 \text{ \lor } A2 \text{ \lor } A3 \text{ \lor } A4 \), then we could split into four subgoals, proving \( B \) in each case of whether \( A1 \), \( A2 \), \( A3 \), or \( A4 \) held. Now \( A \) uses the truth-valued functions \text{ OR } and \text{ =}, rather than the logical connectives \text{ \lor } and \text{ =}, but we have theorems to correct this, using a weaker kind of formula rewriting:

\[
\text{OR_EQ_TT} \quad \text{!p q. p OR q == TT } \Rightarrow \text{ p==TT \text{ \lor } q==TT} \\
\text{EQUAL_TT} \quad \text{!x y. x=y == TT } \Rightarrow \text{ x==y}
\]

\[3\] These three hold vacuously, by contradicting the antecedent \( tb \text{ OCCS } tc \text{ == } TT \).
The inference rule MP_CHAIN, given a list of implications, recursively modifies a theorem using Modus Ponens on it and its parts. With the above theorems, it can change our antecedent to a disjunction of equalities.

```
before:
((tb = t1) OR (tb OCCS t1)) OR
((tb = t2) OR (tb OCCS t2))  == TT

after:
tb==t1  \  tb OCCS t1 == TT  \ /
tb==t2  \  tb OCCS t2 == TT
```

This theorem has the proper form for the tactic SUBST_CASES_TAC. This splits the goal into four cases, and substitutes each equality through the corresponding goal and its assumptions. Figure 4 shows the first two cases; the other two are similar. An asterisk (*) marks those assumptions which, altered by the substitution, match part of the goal. Now ASM_REWRITE_TAC can finish each case.

```
"((ta = t1) OR (ta OCCS t1)) OR ((ta = t2) OR (ta OCCS t2)) == TT"
[ "^ ta == UU" ]
[ "ta OCCS tb == TT" ]
[ "TT == TT ==> ta OCCS t1 == TT" ]
[ "tb OCCS t2 == TT ==> ta OCCS t2 == TT" ]
[ "^ t1 == UU" ]
[ "^ t2 == UU" ]

"((ta = t1) OR (ta OCCS t1)) OR ((ta = t2) OR (ta OCCS t2)) == TT"
[ "^ ta == UU" ]
[ "ta OCCS t1 == TT" ]
[ "t1 OCCS t1 == TT ==> ta OCCS t1 == TT" ]
[ "t1 OCCS t2 == TT ==> ta OCCS t2 == TT" ]
[ "^ t1 == UU" ]
[ "^ t2 == UU" ]
```

Figure 4. Two cases after substitution in the assumptions.
In the goal \( A \Rightarrow B \), how do we grab hold of the antecedent \( A \), to put it through MP\_CHAIN and SUBST\_CASES\_TAC? During the interactive search for a proof, we might use DISCH\_TAC to put \( A \) on the assumption list, then use one of LCF's tacticals for manipulating assumptions. But we have a slick way of expressing the completed proof. Instead of the tactic DISCH\_TAC, we can use the tactical DISCH\_THEN, which binds the antecedent \( A \) to a variable for further use.

\[
\text{TERM\_TAC "to" THEN} \\
\text{ASM\_REWRITE\_TAC [OCCS\_CLAUSES; OCCS\_EQ]} \\
\text{THEN} \\
\% solves all base cases, but the COMB case remains%} \\
\text{DISCH\_THEN } \% binds antecedent to a variable% \\
\text{(\ante. SUBST\_CASES\_TAC (MP\_CHAIN [OR\_EQ\_TT; EQUAL\_TT]) ante))} \\
\text{THEN} \\
\% splits into four cases%} \\
\text{ASM\_REWRITE\_TAC [OR\_CLAUSES; OR\_R\_TT; OR\_TOTAL;} \\
\text{TERM\_EQUAL\_TOTAL; OCCS\_TOTAL]
\]

In the jargon of denotational semantics (Stoy [1977]), the argument to DISCH\_THEN is a continuation that tells what to do with the antecedent. Only time will tell whether such a high-powered approach can be justified; flexibility to try different styles is the hallmark of LCF.

7. Postscript

We can prove many similar theorems. The ordering relation OCCS is anti-reflexive; it is also monotonic with respect to substitution.
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\[ \begin{align*}
&!t. \quad t \text{ OCCS} t == TT \\
&!sl. \quad sl==UU ==> \\
&!t. \quad t==UU ==> \\
&!u. \quad t \text{ OCCS} u == TT ==> \\
&\quad (t \text{ SUBST} sl) \text{ OCCS} (u \text{ SUBST} sl) == TT
\end{align*} \]

Most of these proofs are straight-forward inductions, but some reveal weaknesses in LCF. For instance, we plan to extend the backwards-chaining primitives to handle existential implications such as \((?x.A)==B\). This would be executing PPLAMBDA theorems as a Prolog program (Clocksin and Mellish [1981]).

LCF's methods apply to any logic. The logic PPLAMBDA can complicate first-order problems, adding cases about undefined elements. However, the extra cases are usually trivial. PPLAMBDA is essential for proofs about denotational semantics, compiler correctness, lazy evaluation, and higher-order functional programs.

In such a short paper it is impossible to document, motivate, or even mention all the techniques -- particularly experimental ones. You may not see how LCF helped to discover the proofs shown here, since I have omitted the fruitless first attempts. Look again at the subgoals in Figures 3 and 4, which LCF printed. Imagine writing them out by hand. With computer assistance, we can hope to prove theorems involving increasingly complex data structures.

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