Untyped strictness analysis

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Abstract

We re-express Hudak and Young's higher-order strictness analysis for the untyped \(\lambda\)-calculus in a conceptually simpler and more semantically-based manner. We show our analysis to be a sound abstraction of Hudak and Young's which is also complete in a sense we make precise.

Background

Untyped strictness analysis is currently a little out of vogue. There are two reasons for this. One is that the standard reference [3] is presentationally hard to read and, as we show, is complicated by spurious domain elements. The other is that most of the functional programming world uses some form of typed (typically simple polymorphic) \(\lambda\)-calculus. Strictness analysis for such languages benefits from the simple exposition of the Imperial College stable and various finiteness properties seemingly associated by the decidability of type inference.

However, some properties of (e.g.) the second-order polymorphic \(\lambda\)-calculus are best proved by appeal to untyped results and, as yet, we know of no polymorphic invariance properties which allow lifting of results for simple types.

It is with this interest in such strictness analysis that we give a more fundamental explanation of the ideas in the untyped \(\lambda\)-calculus which both better explains the theory and encourages its use as a basis for such extended analyses.

We discuss the treatment of domain errors which influence strictness. In particular, it is common to wish errors to give non-\(\bot\) values (exceptions) in an untyped language, but when we see the untyped language used as an underlying implementation of a typed language such as the 2nd-order \(\lambda\)-calculus then (unobtainable) domain errors should be treated as \(\bot\) to ease strictness analysis.

1 Introduction

Strictness analysis was originated by Mycroft [8] for the first-order case over flat domains, using a formalism based on abstraction and concretisation functions.
Temporarily, suppose that $D$ is a flat cpo. Let $2$ stand for the set $\{0,1\}$ ordered by $0 < 1$. Recall that $f : D^n \rightarrow D$ is strict in its $k$-th argument if $(\forall x \in D^n) f(x_1, \ldots, x_{k-1}, \bot, x_{k+1}, \ldots, x_n) = \bot$. Mycroft developed a strictness theory for first order functions on flat domains which gave a standard interpretation of a program user-defined function symbol (say $f$) as a function $f$ as above and also non-standard interpretation $f^\sharp : 2^n \rightarrow 2$. Such $f^\sharp$ satisfy a correctness property with respect to $f$ along the lines of $f^\sharp(1, \ldots, 1, 0, 1, \ldots, 1) = 0 \Rightarrow f$ is strict in its $k$-th argument. This property is respected by composition and fixpoint extraction and so lifts from base functions to user-defined functions.

Burn, Hankin and Abramsky [1] showed that the Hoare (or relational) power-domain could be used to generate a theory of strictness analysis for the simply typed $\lambda$-calculus. (Their system abstracts functions between concrete domains with functions between abstract domains).

Around the same time Hudak and Young [3] gave a definition of strictness pairs which enabled them to analyse the untyped $\lambda$-calculus. They observed that an expression has not only a “direct strictness” (the set of variables which are evaluated when it is), but also a “delayed strictness” (the set of variables which are evaluated when the expression is applied). They suggested that the strictness property should perhaps be captured by an object, the domain of strictness pairs $Sp$ defined by:

$$Sp = \mathcal{P}(V) \times (Sp \rightarrow Sp)$$

where $V$ is the set of variable names and $\mathcal{P}(V)$ is ordered by reverse inclusion $\supseteq$. With every expression $e$ in a strictness environment $senv$, they associated a strictness pair that provides properties of $e$ both as an ‘isolated value’ and as a ‘function to be applied’:

$$s[e]senv = (sv, sf)$$

This work was less semantically based than Burn, Hankin and Abramsky’s because its use of power-sets of variable names in the ‘strictness pair’ domain introduced syntactic objects into a semantic construction. In retrospect, it was both over-syntactic and unnecessary in the sense that $\mathcal{P}(V)$ can be replaced by $2$ with no loss of expressive power as we show in section 3, where we use the notation $\mathcal{E}_{HY}[\cdot]$ instead of $\mathcal{S}[\cdot]$.

This work is structured in the following manner. Section 2 explains notation and the syntax and standard semantics for the untyped $\lambda$-calculus. It also describes the problem of domain error. Section 3 gives strictness interpretations which formalise Hudak and Young’s and also our improvement. Section 4 sets up the relationship between the standard semantics, Hudak and Young’s strictness and ours. Section 5 shows the correctness and completeness of our strictness interpretation relative to Hudak and Young’s.

2 Notation and $\lambda$-calculus

Here we use the word domain to mean complete (pointed) partial order as usual. Let $2$ stand for the domain $\{0,1\}$ ordered by $0 < 1$. Recursive domain definitions
are as usual and +, \oplus, \times, \rightarrow will mean respectively separated sum, coalesced sum, cartesian product and continuous function space.

2.1 Untyped \(\lambda\)-calculus

We consider the untyped \(\lambda\)-calculus with constants. Let \(C\) and \(V\) be sets of constants (including primitive functions) and variables ranged over by \(c\) and \(x\) respectively (\(z\) will also be used to range over integer constants). For the purposes of this paper we will assume \(C\) contains \(\mathbb{Z}\) and Turing-sufficient arithmetic constants \{\text{plus}, \text{minus}, \text{cond}\}. (The first argument of \text{cond} is required to be an integer which is tested for zero/non-zero as in the 'C' programming language).

The set \(\Lambda\) of \(\lambda\)-calculus terms is then:

\[
e \in \Lambda ::= c \mid x \mid \lambda x.e \mid e e'
\]

The standard domain of interpretation is:

\[
U = \mathbb{Z} + (U \to U) + \{\text{wrong}\} \equiv \mathbb{Z}_{\bot} \oplus (U \to U)_{\bot} \oplus \{\text{wrong}\}_{\bot}.
\]

Injections into this sum will be written \(\text{in}_x(\cdot), \text{in}_f(\cdot), \text{and in}_w(\cdot)\). We use typewriter font for syntactic objects and italic font for mathematical (meta-language) objects.

In the untyped world we need to inject functions (in \(U \to U\)) into \(U\) to represent them as values and inject them from \(U\) to \(U \to U\) to apply them. This can be summarised by two functions \(\text{lam}\) and \(\text{app}\) respectively such as:

\[
\begin{align*}
\text{lam } x &= \text{in}_f(x) \\
\text{app } x y &= \text{case } x \text{ of } \text{in}_f(f) \Rightarrow f(y) \\
& \quad \text{else err.}
\end{align*}
\]

Here 'err' typically represents \(\bot\) or \(\text{in}_w(\text{wrong})\), see below. Hudak and Young use the symbol 'error' to represent such domain errors for constants — their treatment of these (and also for \(\text{app}\)) suggests they mean our \(\text{in}_w(\text{wrong})\). Milner [7] used a similar 'wrong' value to handle domain errors.

2.2 Definition of an interpretation

An interpretation \(I\) is a tuple \((D_I; \text{lam}_I, \text{app}_I, \text{num}_I, \text{plus}_I, \text{minus}_I, \text{cond}_I, \text{err}_I)\) where \(D_I\) is a cpo and \(\text{lam}_I : (D \to D) \to D\) and \(\text{app}_I : D \to (D \to D)\) are continuous functions; \(\text{num}_I : \mathbb{Z} \to D\) is a function and \(\text{plus}_I, \text{minus}_I, \text{cond}_I, \text{err}_I \in D\). (We drop the subscripts when the context is clear.)

Given such an interpretation, \(I\), we can define the notion of environment (over \(I\)) by

\[\text{Env}_I = V \to D\]

We use the letter \(\rho\) to range over environments. Such an interpretation, \(I\), naturally defines an associated semantics

\[\varepsilon_I : \Lambda \to \text{Env}_I \to D\]
in the following manner:

$$E_I[x]\rho = \rho(x)$$
$$E_I[c]\rho = \mathcal{K}_I[c]$$
$$E_I[\lambda x.e]_\rho = \text{lam}_I(\lambda d \in D.E_I[e]_\rho[d/x])$$
$$E_I[e \ e']_\rho = \text{app}_I(E_I[e]_\rho)(E_I[e']_\rho)$$

Here we use \(\mathcal{K}_I\) for the meaning of constants — it is simply given by

$$\mathcal{K}_I[z] = \text{num}_I(z)$$
$$\mathcal{K}_I[\text{plus}] = \text{plus}_I$$
$$\mathcal{K}_I[\text{minus}] = \text{minus}_I$$
$$\mathcal{K}_I[\text{cond}] = \text{cond}_I.$$  

We write \(STD\) to refer to the standard interpretation given by \(U\) as domain and the constants as given below. Arithmetic constants have the usual meanings for arguments within \(\mathbb{Z}\) in \(STD\) (including \(\text{num} z = in_z z\)) — we now consider their definition over the larger space \(D\). The otherwise unused \(err_{STD}\) provides a convenient way of varying the error value in \(\text{plus}, \text{app}\) etc. used in the semantics for constants. (This is important as strictness depends on it.) Although this is rather an abuse of notation, given an interpretation, say \(STD\) above, we will write \(STD[\bot/err]\) or \(STD[\text{in}_w(\text{wrong})/err]\) to represent an interpretation in which the error value and all parts of the interpretation which use it are altered.

2.3 Semantics of constants

2.3.1 Treatment of domain errors

We use the phrase “domain errors” to refer to situations such as \(\text{plus}(\lambda x.x)3\) or \(3(2)\) in which an inappropriate value is used for an operand. To clarify this, let us consider an example, the function \(F\) defined by

$$F = \lambda x.\lambda y.\text{plus} x y$$

Is \(F\) strict in \(y\)? In the standard interpretation we obviously have

$$\text{plus}(in_z(m))(in_z(n)) = in_z(m + n)$$

but this does not define the other cases of \(\text{plus}(in_f(f))\) and \(\text{plus}(in_z(m))(in_f(g))\). If we define

$$\text{plus}(in_f(f)) = \bot$$

then \(F\) is strict in \(y\), but if we define

$$\text{plus}(in_f(f))y = \text{in}_w(\text{wrong})$$

then \(F\) is non-strict in \(y\). Similarly \(\text{app}(in_z(x)x)\) and \(\text{app}(in_w(\text{wrong}))x\) provide similar choices which affect strictness.

As Mishra noted in [5] some very specific choices are made in the denotational semantics regarding such issues as: domain errors due to primitive functions or whether all looping terms should be regarded as denoting the same value.
2.3.2 Subtlety of partial applications

Note that, even for a fixed choice of domain error value there is still a non-trivial choice for semantics of partially applied constants. Clarifying Hudak and Young's remark, there is a non-trivial choice of semantics of the (strict, curried) constants due to the lifting which occurs as a consequence of the above separated sum. (The problem arises from the non-isomorphism of \((A \times B \rightarrow C)\) and \((A \rightarrow (B \rightarrow C))\) which causes \(\eta\)-equivalence to fail). For example, in the standard interpretation we can give

\[
\mathcal{K}[\text{plus}] = \begin{cases} \text{in}_z \lambda x. \text{in}_y \lambda y. \text{case } (x, y) \text{ of } (\text{in}_z(i), \text{in}_z(j)) \Rightarrow \text{in}_z(i + j) \\ \text{else err} \end{cases}
\]

\[
\mathcal{K}[\text{cond}] = \begin{cases} \text{in}_z \lambda x. \text{in}_y \lambda y. \text{in}_z \lambda z. \text{case } x \text{ of } \text{in}_z(n) \Rightarrow (n \neq 0 \rightarrow y, z) \\ \text{else err} \end{cases}
\]

or we can give the following versions (which are more strict in the case of \(\text{err} = \perp\))

\[
\mathcal{K}[\text{plus}] = \begin{cases} \text{in}_z \lambda x. \text{case } x \text{ of } \text{in}_z(i) \Rightarrow \text{in}_y \lambda y. \text{case } y \text{ of } \text{in}_z(j) \Rightarrow \text{in}_z(i + j) \\ \text{else err} \end{cases}
\]

\[
\mathcal{K}[\text{cond}] = \begin{cases} \text{in}_z \lambda x. \text{case } x \text{ of } \text{in}_z(n) \Rightarrow (n \neq 0 \rightarrow (\text{in}_y \lambda y. \text{in}_z \lambda z. y), (\text{in}_y \lambda y. \text{in}_z \lambda z. z)) \\ \text{else err} \end{cases}
\]

Such differences are important for the precise details of the abstract strictness interpretation given in section 3.

To reproduce as closely as possible Hudak and Young's world, we adopt the former definitions and \(\text{err}_{\text{STD}} = \text{in}_w(\text{wrong})\).

3 Untyped strictness

In this section we give a simpler and more semantically oriented framework for the strictness analysis of Hudak and Young [3]. Section 2 gave the syntax and standard interpretation of our \(\lambda\)-calculus which yields the standard value domain

\[
U = \mathbb{Z} + (U \rightarrow U) + \{\text{wrong}\} \quad [\equiv \mathbb{Z}_\perp \oplus (U \rightarrow U)_\perp \oplus \{\text{wrong}\}_\perp]
\]

Now, since the abstract domain for \(\mathbb{Z}_\perp\) is to be 2 as in the first order case, it might appear that the cpo

\[
S = \{1\} + (S \rightarrow S)
\]

is a suitable domain of strictness properties (the separated sum adds a \(\perp\) element corresponding to 0. However, the untyped nature of functions like \(\lambda x. \text{cond} \times 7 \ (\lambda y. 42 + y)\) means that we need more least upper bounds to exist. Recalling the natural isomorphism of \(\mathcal{P}(A + B)\) and \(\mathcal{P}(A) \times \mathcal{P}(B)\) and the similarly of uncertainty induced by imprecise knowledge and non-determinism leads us to consider the larger cpo given by

\[
S = 2 \times (S \rightarrow S)
\]
which can now be viewed as a simpler formulation of Hudak and Young’s strictness pairs. We adopt the name strictness pairs and their notation: elements \( s \in S \) are written \( (v, f) \) with \( s_v, s_f \) standing for the components of \( s \).

### 3.1 Strictness in the presence of domain errors

Note that the treatment of domain errors affects strictness. In the \( STD[in\,(\text{wrong})/\text{err}] \) interpretation above, we have that \( \lambda x.\text{cond}(\lambda y.x) \ x \ x \) is not strict in \( x \) and hence neither is \( \lambda x.\lambda y.\text{cond} \ y \ x \ x \). Oddly, Hudak and Young’s original analysis incorrectly gives these as strict.

### 3.2 Strictness semantic interpretation

We take

\[
S = 2 \times (S \rightarrow S)
\]

as above for the domain part of the interpretation. Then the interpretation is completed by:

\[
\begin{align*}
\lambda x &= \langle 1, x \rangle \\
\text{app} x y &= \langle x_v \cap (x_f y)_v, (x_f y)_f \rangle \\
&= \langle x_v, T_{S \rightarrow S} \cap (x_f y) \rangle \\
\text{err} &= \langle 1, \lambda s.\text{err} \rangle \\
&= T_S \\
\text{num} z &= \langle 1, \lambda s.\text{err} \rangle \\
\text{plus} &= \text{minus} = \langle 1, \lambda x.\langle 1, \lambda y.(x_v \cap y_v, \lambda s.\text{err}) \rangle \rangle \\
\text{cond} &= \langle 1, \lambda x.\langle 1, \lambda y.\langle 1, \lambda z.\langle x_v \cap (y_v \cup z_v), y_f \cup z_f \rangle \rangle \rangle \rangle \\
&= \langle 1, \lambda x.\langle 1, \lambda y.\langle 1, \lambda z.\langle x_v, T_{S \rightarrow S} \cap (y \cup z) \rangle \rangle \rangle \rangle
\end{align*}
\]

The strictness interpretation of \( \text{cond} \) above is for the first choice (i.e. Hudak and Young’s) of standard semantics of \( \text{plus} \) and \( \text{cond} \) given in section 2.3.2, i.e. when \( \text{cond} \bot \neq \bot \). For the case of \( \text{cond} \bot = \bot \) we would have the better (enabling more strictness inferences) interpretation as

\[
\begin{align*}
\text{plus} &= \langle 1, \lambda x.\langle x_v, \lambda y.(x_v \cap y_v, \lambda s.\text{err}) \rangle \rangle \\
\text{cond} &= \langle 1, \lambda x.\langle x_v, (x_v = 0) \rightarrow \lambda s.\text{err}, \lambda y.(1, \lambda z.(y \cup z)) \rangle \rangle.
\end{align*}
\]

We will refer to this interpretation as \( EM \) and use ‘\( EM \)’ subscripts on its components when the context requires.

### 3.3 Hudak and Young’s strictness interpretation

Let us call HY-strictness the strictness interpretation \( HY \) defined by

\[
(S_HY; \lambda m_HY, \text{app}_H Y, \text{num}_H Y, \text{plus}_H Y, \text{minus}_H Y, \text{cond}_H Y, \text{err}_H Y)
\]

satisfying the definitions below. These are taken from the strictness semantics of Hudak and Young, save that we use the \( \cup \) symbol to denote the least upper bound
on \( S_{HY} \rightarrow S_{HY} \) but inexplicably they use \( \cap \) "for clarity". Similarly, to make the semantic basis clearer, we have used the \( \sqcup \) symbol instead of the synonymous \( \cap \) on \((P(V), \supseteq)\) and similarly \( \cap \) for \( \sqcup \). We also have no need for "hatted" variables \( \hat{\cdot} \) to range over sets of variables which they used because of their mix of syntax and semantics. \( HY \) is given, dropping subscripts, by:

\[
S = (P(V), \supseteq) \times (S \rightarrow S)
\]

\[
lam x = (\{\}, x)
\]

\[
app x y = (x_v \cap (x_f y)_v, (x_f y)_f)
\]

\[
= (x_v, T_{S \rightarrow S} \cap (x_f y))
\]

\[
err = (\{\}, \lambda s. err)
\]

\[
= T_s
\]

\[
num z = (\{\}, \lambda s. err)
\]

\[
plus = minus = (\{\}, \lambda x. (\{\}, \lambda y. (x_v \cap y_v, \lambda s. err)))
\]

\[
cond = (\{\}, \lambda x. (\{\}, \lambda y. (\{\}, \lambda z. (x_v \cap (y_v \cup z_v), y_f \cup z_f))))
\]

\[
= (\{\}, \lambda x. (\{\}, \lambda y. (\{\}, \lambda z. (x_v, T_{S \rightarrow S} \cap (y \cup z)))))
\]

It appears that merely re-phrasing Hudak and Young’s formulation as an interpretation helps to separate syntax and semantics.

### 3.3.1 Warning

As we noted in section 3.1 the definition of \( cond_{HY} \) is only correct with respect to \( err_{STD} = \bot \) not \( err_{STD} = in_w(wrong) \). Accordingly, to ensure the correctness of the following theorem from now on we take

\[
K_{STD}[cond] = in_f \lambda x. in_f \lambda y. in_f \lambda z. case x of \ in_f(n) \Rightarrow (n \neq 0 \rightarrow y, z)
\]

\[
\text{else } \bot
\]

instead of that given in section 2.3.2.

### 4 Relationship between various interpretations

We claim the following results.

1. (From Hudak and Young) \( HY (= HY[T_{S_{HY}}/err]) \) is a correct abstraction of \( STD (= STD[in_w(wrong)]/err) \)

2. \( HY[\bot/err] \) is a correct abstraction of \( STD[\bot/err] \)

3. \( EM \) is a correct abstraction of \( HY \)

4. \( EM \) is complete for \( HY \)

5. \( EM[\bot/err] \) is a correct abstraction of \( HY[\bot/err] \)

6. \( EM[\bot/err] \) is complete for \( HY[\bot/err] \)

The correctness relations between \( STD \) and \( EM \) hold by transitivity.

The next section sets about proving that results 3 and 4, i.e. that \( EM \) is a correct abstraction of \( HY \) which is also complete.
5 Relationship to Hudak and Young’s strictness

We now set up a relationship between between HY-strictness $HY$ and EM-strictness $EM$ from sections 3.2 and 3.3. This relationship is then shown to induce an abstraction of HY-strictness into EM-strictness. Moreover, the abstraction is complete in that all properties exploited by Hudak and Young are derivable via our strictness interpretation.

For notational reasons in this section we will use $A$ for $S_{EM}$ and $B$ for $S_{HY}$.

Both $A$ and $B$ are given as recursive function spaces, viz

$$A = 2 \times (A \to A)$$
$$B = (P(V), \supseteq) \times (B \to B)$$

Let us define $\gamma_1 : 2 \to P(V)$ by

$$\gamma_1(0) = V$$
$$\gamma_1(1) = \{\}.$$ 

Now, the relation we seek to define should satisfy

$$\sim \subseteq A \times B$$

$$(x, f) \sim (y, g) \iff (\forall a \in A, b \in B) a \sim b \Rightarrow f(a) \sim g(b)$$

but it is unclear whether this is a well-definition. To prove the unique existence and various properties of $\sim$ we define it simultaneously with the inverse limit construction for $A$ and $B$.

Recall that domain equations like that for $A$ above are solved by the inverse limit construction — we put $A_0 = \{\bot\}$, the trivial domain, and then put $A_{k+1} = 2 \times (A_k \to A_k)$. There are embedding $i_k : A_k \to A_{k+1}$ and projection $p_k : A_{k+1} \to A_k$ maps between $A_k$ and $A_{k+1}$. $A$ is obtained as the limit

$$A_\infty = \{(a_0, a_1, \ldots) \in \prod_k A_k \mid a_k = p_{k+1}(a_{k+1})\}$$

The isomorphism of $A$ and $2 \times (A \to A)$ is obtained pointwise from the $p_k$ and $i_k$.

The construction for $B$ is identical.

We can define approximants of $\sim$ in the following manner

$$\sim_k \subseteq A_k \times B_k$$

$$a \sim_0 b \triangleq \text{true}$$

$$(x, f) \sim_{k+1} (y, g) \triangleq y = \gamma_1(x) \land$$

$$(\forall a \in A_k, b \in B_k) a \sim_k b \Rightarrow f(a) \sim_k g(b)$$

and hence properly define

$$\sim \subseteq A \times B$$

$$(a_0, a_1, \ldots) \sim (b_0, b_1, \ldots) \iff (\forall k) a_k \sim_k b_k.$$
It is convenient to write
\[
\downarrow \subseteq 2 \times \mathcal{P}(V) \\
\sim_{k+1} \subseteq (A_k \rightarrow A_k) \times (B_k \rightarrow B_k) \\
x \downarrow y \iff y = \gamma_1(x) \\
f \sim_{k+1} g \iff (\forall a \in A_k, b \in B_k) \ a \sim_k b \Rightarrow f(a) \sim_k g(b)
\]
so that
\[
(x, f) \sim_{k+1} (y, g) \iff x \downarrow y \land f \sim_{k+1} g.
\]

It is also convenient to define here the type-induced ('logical') relations from \(\sim\). Allowing \(t\) to range over meta-language types given by \(t ::= D | t \rightarrow t\) we define
\[
a \sim_D b \iff a \sim b \\
f \sim_{t\rightarrow t'} g \iff ((\forall x,y) \ x \sim^t y \Rightarrow f(x) \sim^{t'} g(y))
\]
The limit relation \(\sim\) now coincides with \(\sim_{D\rightarrow D}\)

We now have distributivity lemma for \(\sim\):

**Lemma:** \(\sim\) preserves arbitrary LUBs and GLBs (including \(\perp\) and \(\top\)) in that, given possible empty sequences \(a_i^i \in A, b_i^i \in B\), we have
\[
((\forall i) \ a^i \sim b^i) (\bigsqcup_i a^i \sim \bigsqcup_i b^i) \land (\forall i) a^i \sim (\forall i) b^i)
\]

### 5.1 Proposition: relatedness

For all \(\lambda\)-terms \(e \in \Lambda\) we have that
\[
(\forall \eta \in Env_{EM}, \rho \in Env_{HY}) \ \eta \sim \rho \Rightarrow E_{EM}[e] \eta \sim E_{HY}[e] \rho
\]
where \(\eta \sim \rho \Rightarrow (\forall x \in V) \ \eta(x) \sim \rho(x)\). It turns out that this abstraction relation is both correct and complete and we study these aspects after a proof sketch.

### 5.2 Proof

We the above proposition by structural induction on the (object) term \(e\). But first we need some lemmas, *viz*

- \(app_{EM} \sim_{D\rightarrow(D\rightarrow D)} app_{HY}\)
- \(lam_{EM} \sim_{(D\rightarrow D)\rightarrow D} lam_{HY}\).
- \((\forall z \in \mathbb{Z}) num_{EM}(z) \sim num_{HY}(z)\)
- \(plus_{EM} \sim plus_{HY}\)
- \(minus_{EM} \sim minus_{HY}\)
• \( \text{cond}_{EM} \sim \text{cond}_{HY} \)
• \( \text{err}_{EM} \sim \text{err}_{HY} \)

Given these lemmas, proved below, the theorem is a trivial structural induction.

We give two cases:
• case \( e = x \): trivial.
• case \( e = \lambda x. e' \): By inductive hypothesis, supposing also \( a \sim b \) then \( \text{E}_{EM}[e'][\eta[a/x]] \sim \text{E}_{HY}[e'][\rho[b/x]] \). Hence by the lemma \( \text{lam}_{EM}\lambda.\text{E}_{EM}[e'][\eta[a/x]] \sim \text{lam}_{EM}\lambda.\text{E}_{HY}[e'][\rho[b/x]] \).

Proof of lemmas

We give the representative cases for \( \text{app} \) and \( \text{cond} \).
• \( \text{app}_{EM} \sim D \rightarrow (D \rightarrow D) \) \( \text{app}_{HY} \): Assume \( a \sim b \) and \( a' \sim b' \) then, expanding the definitions of \( \text{app}_{HY} \) and \( \text{app}_{EM} \), it is equivalent to prove
  \[
  \langle a_v, T_{A\rightarrow A} \rangle \cap \langle a, T_{B\rightarrow B} \rangle \sim \langle b_v, T_{B\rightarrow B} \rangle \cap \langle b, T_{B\rightarrow B} \rangle.
  \]
This holds since \( a \sim b \Leftrightarrow a_v \parallel b_v \land a \parallel b \) and the lemma for \( \sim \)-preservation of \( \cap \) and \( \cap \).
• \( \text{cond}_{HY} \sim \text{cond}_{EM} \): We need to prove
  \[
  \langle 1, \lambda a. \langle 1, \lambda a'. \langle 1, \lambda a''. (a_v, T_{A\rightarrow A}) \cap (a' \cup a'') \rangle \rangle \sim \langle \{\}, \lambda b. \langle \{\}, \lambda b'. \langle \{\}, \lambda b''. (b_v, T_{B\rightarrow B}) \cap (b' \cup b'') \rangle \rangle \rangle.
  \]
Assume \( a \sim b \), \( a' \sim b' \) and \( a'' \sim b'' \) then, using the recursive definition of \( \sim \) and recalling that \( 1 = T_A \) and \( \{\} = T_B \), this is equivalent to
  \[
  T_A \vdash T_B \land \langle a_v, T_{A\rightarrow A} \rangle \cap (a' \cup a'') \sim \langle b_v, T_{B\rightarrow B} \rangle \cap (b' \cup b'').
  \]
The first conjunct holds by definition and by the lemma for \( \sim \)-preservation of \( \cup \) and \( \cap \) it suffices to show
  \[
  \langle a_v, T_{A\rightarrow A} \rangle \sim \langle b_v, T_{B\rightarrow B} \rangle.
  \]
This holds since \( a \sim b \Rightarrow a_v \land b_v \) and
  \[
  T_{A\rightarrow A} = \lambda x. A \backslash T_A \models \lambda y. B \models T_B = T_{B\rightarrow B}.
  \]

5.3 Proposition: soundness

The relation \( \sim \) restricts to an embedding-closure pair (an abstraction of \( B = S_{HY} \n \) by \( A = S_{EM} \)). The concretion and abstraction maps respectively are \( \gamma : A \rightarrow B \) and \( \alpha : B \rightarrow A \) given by
  \[
  \gamma(a) = \{b \in B | a \sim b\}
  \alpha(b) = \cap\{a \in A | a \sim b\} = \cap\{a \in A | \gamma(b) \subseteq a\}
  \]
The \( \alpha \) and \( \gamma \) form a galois connection as usual and correctness of the remainder of the interpretation interpretation (i.e. \( \text{lam}, \text{app}, \text{plus} \) etc.) with respect to \( (\alpha, \gamma) \) follows from that the base lemmas above.
5.4 Proposition: completeness

Since the trivial abstract interpretation would be sound with respect to HY-strictness, we now show that EM-strictness can provide all the information that HY-strictness can. This is a completeness argument. Note that we cannot expect to have a natural completeness result of the form "EM-strictness of expressions determines their HY-strictness". Consider the term $\lambda x.x$: this has HY-strictness of $(\{\}, \lambda x \in S_{HY}.x)$ and EM-strictness of $(0, \lambda x \in S_{EM}.x)$. It is unreasonable to expect some function of the latter, coarser-grained, interpretation to yield the former, finer, one.\(^1\)

Accordingly, our completeness result relies on the observation that Hudak and Young's analysis makes strictness optimisations only on the basis of limited predicates (actually whether the first component of $S_{HY}$ is empty or non-empty). The rest of the internal structure is non-observable. Accordingly, we wish to assert that our simpler internal structure gives rise to the precisely the same observable properties.

The key notion is that both the EM and HY interpretations are only used for strictness optimisations, i.e. early evaluation of an expression. Although it is rarely clearly stated, we implicitly have a predicate which whose result tells us when an abstract value permits strictness optimisations. Here, this predicate (subset of $S_{HY}$ or $S_{EM}$) is given by

$$p(v, f) \iff v = \bot.$$  

This is a sound predictor of when the standard interpretation gives $\bot$ for some prescribed assignments of values to free variables. We abuse notation by using the $p$ for both $S_{HY}$ and $S_{EM}$.

Our completeness result is that, for all meta-terms $e$,

$$(\forall \eta \in Env_{EM}, \rho \in Env_{HY}) \eta \sim \rho \Rightarrow (p(\varepsilon_{EM}[e][\eta]) \iff p(\varepsilon_{HY}[e][\rho]))$$

Thus all optimisations permitted by the HY interpretation are also permitted by the EM interpretation. This forms the basis of our claim that the HY domain had spurious elements.

6 Problem of infinite chains

Hudak and Young mentioned in [3] that their higher-order analysis is not guaranteed to terminate. Indeed, this is the case when a strictness pair needs to be applied an infinite number of times. They gave the following example, $f = \lambda x. f \ x \ x$ which leads to EM-strictness

$$s = \{1, \lambda x. (s_v \cap (s_f x)_v \cap ((s_v x)_v f x)_v, ((s_f x)_f x)_f)\}$$

or HY-strictness

$$s = \{\{\}, \lambda x. (s_v \cup (s_f x)_v \cup ((s_v x)_v f x)_v, ((s_f x)_f x)_f)\}$$

\(^1\)The general question of completeness in abstract interpretation is being developed in a companion paper.
There is a circularity which Hudak and Young suggest is due to the fact that “early” elements of $P(V)$ in the nested pairs depend on “deeper” $S_{HY} \to S_{HY}$ elements. Their solution to this problem of infinite chains is merely suggesting “to impose a weak type discipline”. The next paragraph shows how this could work for the simply typed $\lambda$-calculus and, although this is clearly not the best way to handle the simply typed $\lambda$-calculus, it points to how one might treat the 2nd order $\lambda$-calculus.

Further work

It would be desirable to consider whether certain finite-height lattices could represent strictness properties for the untyped $\lambda$-calculus instead of the infinite chains present in Hudak and Young. For example, if the given program in $\Lambda$ can be corresponds to a (type-stripped) program in the simply typed $\lambda$-calculus (with (object) types ranged over by $t$) then we can use $\sum \mathcal{T}_{\mathcal{Z}_1}[t]$ for the value domain (a retract of $D = \mathcal{Z} + (D \to D)$) and hence $\sum \mathcal{T}_2[t]$ for the strictness domain (a retract of $S = 2 \times (S \to S)$) where

$$\mathcal{T}_X[\text{int}] = X$$
$$\mathcal{T}_X[t \to t'] = \mathcal{T}_X[t] \to \mathcal{T}_X[t'].$$

This exhibits [1] within our model and the key point is that $\sum \mathcal{T}_2[t]$ has no infinite ascending chains. The key question is whether there exists finite height models for another subset of $\Lambda$, those programs corresponding to second-order polymorphically typable terms — this would enable us to conclude Hudak and Young’s suggestion of modelling list operators as $\lambda$-terms and thereby inheriting a sensible strictness theory.

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References


