8 Information Theory (rkh23)

(*a*) Compute the differential Entropy, $H(X)$, for $X \sim U[0.0, 2.0]$ and $X \sim U[0.0, 0.5]$. Provide an argument that differential entropy has meaning, based on the physical interpretation of these values. [4 marks]

(*b*) Consider the Gaussian channel $Y = X + Z$ where X is the input signal with power constraint $E[X^2] \leq P$, and Z is noise-independent of X, with $Z \sim \mathcal{N}(0, N)$.

(*i*) Express the Mutual Information $I(X;Y)$ of this channel in terms of the differential entropies $H(Y)$ and $H(Z)$. [2 marks]

(*ii*) Explain why achieving the capacity of this channel requires X to have a Gaussian distribution. You may assume without proof that the Gaussian distribution maximises differential entropy for a fixed variance. [3 marks]

(*iii*) Derive the Shannon-Hartley theorem for this channel. You may assume the capacity of the Gaussian channel is $C = \frac{1}{2}\log_2\left(1 + \frac{P}{N}\right)$. [3 marks]

(*c*) Consider sending information over two parallel, independent Gaussian channels with noise variances N_1 and N_2 respectively. The transmitter has a total power budget P, which must be distributed between the two channels ($P = P_1 + P_2$).

(*i*) Show that for an optimal solution where both channels are used, the following holds:

$$P_1 + N_1 = P_2 + N_2$$

You may assume the total capacity is the sum of the individual capacities: $C_{total} = \frac{1}{2}\log\_2\left(1 + \frac{P_1}{N_1}\right) + \frac{1}{2}\log\_2\left(1 + \frac{P_2}{N_2}\right)$. [4 marks]

(*ii*) A specific system has $N_1 = 2$ W and $N_2 = 10$ W and $P = 6$ W. Demonstrate that the optimal power allocation cannot be calculated using the relation derived in Part $(c)(i)$, explain why, and state the optimal power distribution. [4 marks]