## 6 Denotational Semantics (im496)

(*a*) Define *adequacy* and *full abstraction* for a semantics of PCF. [4 marks]

(*b*) Consider the flat domains $D = \mathbb{B}_\perp$ and $D = \mathbb{N}_\perp$. Show that the equality function $\mathrm{eq}_D \colon D \times D \to \mathbb{B}_\perp$ determined by

$$\mathrm{eq}_D\ x\ y = \begin{cases} \perp, & \text{if } x = \perp \text{ or } y = \perp \\ \mathrm{true}, & \text{if } x \neq \perp,\ y \neq \perp \text{ and } x = y \\ \mathrm{false}, & \text{if } x \neq \perp,\ y \neq \perp \text{ and } x \neq y \end{cases}$$

is definable in PCF by exhibiting terms $\mathsf{eq}_{\mathsf{bool}}$ and $\mathsf{eq}_{\mathsf{nat}}$ with denotation $[\![\mathsf{eq}_{\mathsf{bool}}]\!] = \mathrm{eq}_{\mathbb{B}_\perp}$ and $[\![\mathsf{eq}_{\mathsf{nat}}]\!] = \mathrm{eq}_{\mathbb{N}_\perp}$. Justify your answer. [5 marks]

(*c*) Consider the *parallel conditional* operations for $D = \mathbb{B}_\perp$ and for $D = \mathbb{N}_\perp$:

$$\mathrm{pif}_D \colon \mathbb{B}_\perp \times D \times D \to D$$

These are the unique continuous functions determined by the equations

$$\mathrm{pif}_D\ \mathrm{true}\ x\ y = x \qquad \mathrm{pif}_D\ \mathrm{false}\ x\ y = y \qquad \mathrm{pif}_D\ \perp\ x\ x = x$$

(*i*) What is the value of $\mathrm{pif}_D \perp x\ y$ for $x \neq y$? Justify your answer. [5 marks]

(*ii*) Prove that $\mathrm{pif}_{\mathbb{B}_\perp}$ is not definable in PCF. You may use the theorem that the *parallel or* operation $\mathrm{por} \colon \mathbb{B}_\perp \times \mathbb{B}_\perp \to \mathbb{B}_\perp$ is not definable. [3 marks]

(*iii*) Prove that $\mathrm{pif}_{\mathbb{N}_\perp}$ is not definable in PCF. [3 marks]