5 Denotational Semantics (im496)

(*a*) Let $(D, \sqsubseteq)$ be a poset and let $f\colon D \to D$ be a monotone function.

(*i*) Define the least pre-fixed point of f. [2 marks]

(*ii*) Show that it exists when D is a domain and f is continuous. [5 marks]

(*b*) The *binary join* of a pair $x, y \in D$ is the least upper bound of the set $\{x, y\}$. In other words, it is an element $(x \sqcup y) \in D$ such that

- $x \leq (x \sqcup y)$ and $y \leq (x \sqcup y)$
- for all $z \in D$, if $x \leq z$ and $y \leq z$, then $(x \sqcup y) \leq z$,

Let D be a poset with binary joins and $x \in D$, and consider the function

$$\mathrm{join}_x\colon D \to D \qquad\qquad \mathrm{join}_x(y) = x \sqcup y$$

(*i*) Show that join_x is monotone. [3 marks]

(*ii*) Show that join_x is continuous when D is a cpo. [5 marks]

(*c*) Suppose that D is a domain with binary joins, $x \in D$, and $f\colon D \to D$ is a continuous function. Show that there exists an element $y \in D$ such that

- $x \sqsubseteq y$ and $f(y) \sqsubseteq y$
- for every $z \in D$, if $x \sqsubseteq z$ and $f(z) \sqsubseteq z$, then $y \sqsubseteq z$.

[*Hint:* Is y a pre-fixed point of some function?] [5 marks]