9 Discrete Mathematics (js2878)

(*a*) Let A be a set and let $R \subseteq A \times A$ be a binary relation on A.

(*i*) **Give** a system of *rules* generating the smallest reflexive relation containing R, so that there exists a derivation of (x, y) if and only if $(x, y) \in \mathbf{1}_A \cup R$. [1 mark]

(*ii*) Add **just one** rule to those of Part (*a*)(*i*) so that the combined system generates the *equivalence closure* of R, *i.e.* the smallest equivalence relation containing R. Argue for the correctness of your rule. [2 marks]

(*b*) Let Σ_0 and Σ_1 be two alphabets, and let $\rho\colon \Sigma_0 \to \Sigma_1$ be a function between them, writing $\rho\colon \Sigma_0^\star \to \Sigma_1^\star$ also for the induced function on strings. Given a formal language $\mathcal{L}$ over Σ_1, we define a formal language $\rho^*\mathcal{L}$ over Σ_0 as follows:

$$\rho^*\mathcal{L} := \{u \in \Sigma_0^\star \mid \rho(u) \in \mathcal{L}\}$$

(*i*) **Show** that if $\mathcal{L}$ is regular, then $\rho^*\mathcal{L}$ is regular too. [4 marks]

(*ii*) Assuming $\rho^*\mathcal{L}$ is regular, **propose and prove** a sufficient condition on ρ for $\mathcal{L}$ to be regular. [4 marks]

(*c*) (*i*) Let $\mathcal{K}$ and $\mathcal{L}$ be regular languages over an alphabet Σ. Say whether or not the formal language

$$\mathcal{M} = \{uv \mid u \in \mathcal{K} \implies v \in \mathcal{L}\}$$

is regular, and provide **proof**. [4 marks]

(*ii*) Let Σ be an alphabet. Give a **necessary and sufficient** condition for the formal language

$$\mathcal{L} = \{a^n b a^n \mid n \text{ even}, a \in \Sigma, b \in \Sigma\}$$

to be regular, and **provide proof**. [5 marks]