8 Discrete Mathematics (js2878)

(*a*) We define the ***kernel*** of a function $f\colon A \to B$ to be the binary relation

$$\ker(f) := \{(x, y) \in A \times A \mid f(x) = f(y)\}.$$

(*i*) **Prove** that $\ker(f)$ is an equivalence relation. [3 marks]

(*ii*) **Prove** that f is an injective function iff $\ker(f)$ is the diagonal relation $\mathbf{1}_A = \{(a, a) \mid a \in A\}$. [1 mark]

(*b*) We define the ***coimage*** of $f\colon A \to B$ to be the quotient

$$\operatorname{coim}(f) := A/\ker(f)$$

of A by the kernel of f. We write $e_f\colon A \twoheadrightarrow \operatorname{coim}(f)$ for the corresponding quotient map. In what follows, we write $i_f\colon \operatorname{im}(f) \hookrightarrow B$ for the inclusion of the *image* of f into B.

(*i*) **Define** a function $h_f\colon \operatorname{coim}(f) \to \operatorname{im}(f)$ and **verify** that the composition

$$A \xrightarrow{e_f} \operatorname{coim}(f) \xrightarrow{h_f} \operatorname{im}(f) \xrightarrow{i_f} B$$

is equal to $f\colon A \to B$. [2 marks]

(*ii*) **Prove** that h_f from Part (*b*)(*i*) is surjective. [2 marks]

(*iii*) **Prove** that h_f from Part (*b*)(*i*) is injective. [2 marks]

(*iv*) **Prove** that f is injective iff $e_f\colon A \twoheadrightarrow \operatorname{coim}(f)$ is injective. [2 marks]

(*c*) Say whether or not the following are true, with proof.

(*i*) If A is finite and B is enumerable, then the function space $A \to B$ is enumerable. [2 marks]

(*ii*) If A and B are countable, then so is the function space $A \to B$. [2 marks]

(*iii*) If A is enumerable, then the quotient A/E of A by any equivalence relation $E \subseteq A \times A$ is enumerable. [2 marks]

(*iv*) If $B(a)$ is an enumerable set for each $a \in A$ and A is enumerable, then the indexed disjoint union $\biguplus_{a \in A} B(a)$ is enumerable. [2 marks]