7 Discrete Mathematics (mpf23)

(*a*) For $\mathbb{N}^+$ the set of positive natural numbers, let

$$S = \left\{ m \in \mathbb{N}^+ \mid \sum_{i=1}^{m-1} i^m \equiv 0 \pmod{m} \right\}$$

where, by convention, $\sum_{i=1}^{0} i = 0$.

(*i*) State whether or not the set S is finite and prove your claim. [4 marks]

(*ii*) For O the set of odd integers, give an explicit description of the intersection set $O \cap S$ and prove it correct. [4 marks]

(*b*) Let $P(n)$ be the property of natural numbers n given by

$$\forall\, (a_1, \ldots, a_n) \in \mathbb{N}^n.\; n^n \cdot \prod_{i=1}^{n} a_i \leq \left(\sum_{i=1}^{n} a_i \right)^n$$

where, by convention, $\prod_{i=1}^{0} a_i = 1$ and $\sum_{i=1}^{0} a_i = 0$.

(*i*) Prove that $P(n)$ holds for all n that are even powers of 2. [6 marks]

(*ii*) Prove that $\forall\, n \in \mathbb{N}.\; P(n+1) \Rightarrow P(n)$. [4 marks]

[Hint: In proving $P(n)$ for the instance $(a_1, \ldots, a_n) \in \mathbb{N}^n$ consider $P(n+1)$ for the instance $\left(n \cdot a_1, \ldots, n \cdot a_n, \sum_{i=1}^{n} a_i\right) \in \mathbb{N}^{n+1}$.]

(*iii*) Argue whether or not $P(n)$ holds for all natural numbers n. [2 marks]