**6 Discrete Mathematics (mpf23)**

For a natural number $n$, define the following sets:

$$[n] = \{\, i \in \mathbb{N} \,:\, 0 \leq i < n \,\} \qquad , \quad [n]^+ = \{\, j \in [n] \,:\, j \neq 0 \,\}$$

$$\mathrm{C}(n) = \{\, k \in [\,n\,]^+ : \gcd(k,n) = 1 \,\} \ , \ \ \mathrm{I}(m,n) = \{\, \iota \in \big([m] \Rightarrow [n]\big) : \iota \text{ is injective} \,\}$$

($a$) Describe the sets $[3] \times [3]^+$, $\mathrm{C}(3^2)$, and $\mathrm{I}(2,3)$ by enumerating their elements. [3 marks]

($b$) For a prime number $p$,

($i$) give a mapping $f_p$ from $\mathrm{C}(p^2)$ to $[p] \times [p]^+$ and prove that it defines a function $f_p : \mathrm{C}(p^2) \to [p] \times [p]^+$; [4 marks]

($ii$) give a mapping $g_p$ from $[p] \times [p]^+$ to $\mathrm{C}(p^2)$ and prove that it defines a function $g_p : [p] \times [p]^+ \to \mathrm{C}(p^2)$; [4 marks]

($iii$) prove that $g_p \circ f_p = \mathrm{id}_{\mathrm{C}(p^2)}$ and that $f_p \circ g_p = \mathrm{id}_{[p]\times[p]^+}$. [2 marks]

($c$) For a prime number $p$, define a function $u_p : \mathrm{C}(p^2) \to \mathrm{I}(2,p)$ and a function $v_p : \mathrm{I}(2,p) \to \mathrm{C}(p^2)$ and prove that they determine bijections between $\mathrm{C}(p^2)$ and $\mathrm{I}(2,p)$. [5 marks]

($d$) For a prime number p, determine the cardinality of $\mathrm{C}(p^2)$. [2 marks]