COMPUTER SCIENCE TRIPOS Part II – 2025 – Paper 9

12 Randomised Algorithms (tms41)

In the weighted vertex cover problem, we are given an undirected graph G = (V, E), n = |V| with a weight function $w : V \to \mathbb{R}^+$. We seek a subset $C \subseteq V$ such that each edge has at least one endpoint in C and $w(C) = \sum_{v \in C} w(v)$ is minimised.

(a) Express the weighted vertex cover problem as an integer program. [3 marks]

Consider the following deterministic greedy algorithm.

Input: $G = (V, E), w : V \to \mathbb{R}^+$ with edges e_1, e_2, \ldots, e_m

• $C = \emptyset$

- For i = 1 to m do
- Let $e_i = \{u, v\}$
- If $\{u, v\} \cap C = \emptyset$ then
- If w(u) < w(v) then $C \leftarrow C \cup \{u\}$ else $C \leftarrow C \cup \{v\}$

(b) Show that the approximation ratio of this algorithm is unbounded in n.

[4 marks]

Consider now the following randomised algorithm.

Input:
$$G = (V, E), w : V \to \mathbb{R}^+$$
 with edges e_1, e_2, \ldots, e_m

•
$$C = \emptyset$$

- For i = 1 to m do
- Let $e_i = \{u, v\}$
- If $\{u, v\} \cap C = \emptyset$ then

With prob.
$$\frac{w(v)}{w(u)+w(v)}$$
 update $C \leftarrow C \cup \{u\}$, otherwise $C \leftarrow C \cup \{v\}$

- (c) Assuming that the above is a randomised 2-approximation algorithm, design a new algorithm such that the returned cover C and optimal cover C^* satisfy $\mathbf{P}[w(C) \leq 3 \cdot w(C^*)] \geq 1 - n^{-1}$. [4 marks]
- (d) Consider the following simplified input. For any integer $k \ge 0$, the star graph $G_k = (V, E)$ is given by $V = \{v, u_1, u_2, \ldots, u_k\}$ and $E = \{\{v, u_i\}: 1 \le i \le k\}$; thus v is a "center" node connected to all other nodes. Let C_k be the returned solution of the randomised algorithm on G_k .
 - (i) Prove that

$$\mathbf{E}\left[w(C_k)\right] \le 2 \cdot w(v).$$

Hint: Find a recursive formula for $\mathbf{E}[w(C_k)]$ and then use induction over k. [5 marks]

(ii) Using the result from (d), prove that the randomised algorithm is a randomised 2-approximation algorithm for any input graph.
Hint: Decompose the edge-set E into star graphs. [4 marks]