4 Denotational Semantics (mgapb2)

In your answers, you are allowed to use theorems from the course, provided you state them precisely beforehand.

Let $(P, \sqsubseteq)$ be a poset. We say a subset $S \subseteq P$ is

- a downset if whenever $y \in S$ and $x \sqsubseteq y$ then also $x \in S$;
- chain-closed if for any chain $x_0 \sqsubseteq x_1 \sqsubseteq \cdots \in S$, $\bigsqcup_i x_i \in S$ whenever the lub exists (in P).

We write $\mathcal{D}(P)$ (respectively $\mathcal{C}(P)$) for the set of downsets (resp. chain-closed downsets) of P, and $\mathcal{P}(S)$ for the powerset of a set S. Given a function $f \in X \to Y$, we write $f^{-1} \in \mathcal{P}(Y) \to \mathcal{P}(X)$ for the inverse image function, which maps a subset $S \subseteq Y$ to $f^{-1}(S) = \{x \in X \mid f(x) \in S\}$.

(*a*) Show that for any set X, $(\mathcal{P}(X), \subseteq)$ is a domain. [4 marks]

(*b*) Show that given any two sets X and Y and a function $f \in X \to Y$, f^{-1} is a strict continuous function. [4 marks]

(*c*) Given two posets P and Q, show that if a function $f \in P \to Q$ is monotone then for all downsets $D \in \mathcal{D}(Q)$, $f^{-1}(D)$ is a downset. [3 marks]

(*d*) Show the converse: if P and Q are two posets, and $f \in P \to Q$ a function such that f^{-1} maps downsets to downsets, then f is monotone. [*Hint:* You might want to consider $\downarrow a$, the set $\{x | x \sqsubseteq a\}$ of elements smaller than a.] [4 marks]

(*e*) Given two chain complete partial orders P and Q, show that a monotone function $f \in P \to Q$ is continuous if and only if f^{-1} maps chain-closed downsets to chain-closed downsets. [5 marks]