8 Discrete Mathematics (mpf23)

(*a*) (*i*) Prove that for all positive integers m and integers n,

$$\gcd(n, m) = m \iff n \equiv 0 \pmod{m}$$

[2 marks]

(*ii*) Without using the Fundamental Theorem of Arithmetic, prove that for all prime numbers p and positive integers n,

$$\gcd(n, p) = 1 \iff n^{(p^2-1)} \equiv 1 \pmod{p}$$

[6 marks]

(*b*) Prove that the sum of the cubes of any three consecutive natural numbers is divisible by 9. [6 marks]

(*c*) Let $\mathcal{U} \subseteq \mathcal{P}(U)$ be a family of subsets of a set U such that, for all $\mathcal{F} \subseteq \mathcal{U}$, the big intersection $\bigcap \mathcal{F}$ is in $\mathcal{U}$.

Prove that, for all $\mathcal{F} \subseteq \mathcal{U}$, there exists a smallest element $\widehat{\mathcal{F}}$ of $\mathcal{U}$ that contains every element of $\mathcal{F}$ as a subset; that is,

(*i*) $\forall\, Y \in \mathcal{U}.\ (\forall\, X \in \mathcal{F}.\ X \subseteq Y\,) \implies \widehat{\mathcal{F}} \subseteq Y$

(*ii*) $\forall\, Z \in \mathcal{F}.\ Z \subseteq \widehat{\mathcal{F}}$

[6 marks]