5 Denotational Semantics (mgapb2)

In all parts of this question, you are allowed to use theorems from the course, provided you state them precisely beforehand. You may also extend a proof by (rule) induction from the course with new cases without reproving the ones from the course, again provided you clearly state the proof you are extending.

(*a*) Given domains D_1, D_2, E_1 and E_2, and continuous functions $f_1 : D_1 \to E_1$ and $f_2 : D_2 \to E_2$, show that

$$\begin{array}{rccc} f_1 \times f_2 : & D_1 \times D_2 & \to & E_1 \times E_2 \\ & (d_1, d_2) & \mapsto & (f_1(d_1), f_2(d_2)) \end{array}$$

is continuous. [6 marks]

We wish to extend PCF with the product type $\tau_1 * \tau_2$, by adding the new terms `fst`, `snd` and `pair` to the language, such that

$$\frac{\Gamma \vdash t : \tau_1 * \tau_2}{\Gamma \vdash \mathtt{fst}(t) : \tau_1} \qquad \frac{\Gamma \vdash t : \tau_1 * \tau_2}{\Gamma \vdash \mathtt{snd}(t) : \tau_2} \qquad \frac{\Gamma \vdash t_1 : \tau_1 \quad \Gamma \vdash t_2 : \tau_2}{\Gamma \vdash \mathtt{pair}(t_1, t_2) : \tau_1 * \tau_2}$$

and with the following operational semantics:

$$\frac{t \Downarrow_{\tau_1 * \tau_2} \mathtt{pair}(v_1, v_2)}{\mathtt{fst}(t) \Downarrow_{\tau_1} v_1} \qquad \frac{t \Downarrow_{\tau_1 * \tau_2} \mathtt{pair}(v_1, v_2)}{\mathtt{snd}(t) \Downarrow_{\tau_2} v_2} \qquad \frac{t_1 \Downarrow_{\tau_1} v_1 \quad t_2 \Downarrow_{\tau_2} v_2}{\mathtt{pair}(t_1, t_2) \Downarrow_{\tau_1 * \tau_2} \mathtt{pair}(v_1, v_2)}$$

(*b*) Give a denotational semantics for the product type $[\![\tau_1 * \tau_2]\!]$ in terms of $[\![\tau_1]\!]$ and $[\![\tau_2]\!]$. [2 marks]

(*c*) Give a denotational semantics for `fst`, `snd` and `pair`, extending the semantics of PCF from the lectures, and justify why this semantics is well-defined according to the typing rules given above. [6 marks]

(*d*) Recall what it means for denotational semantics to be sound. Show that the semantics you have just given is sound. [6 marks]