5 Denotational Semantics (mgapb2)

In all parts of this question, you are allowed to use theorems from the course, provided you state them precisely beforehand. You may also extend a proof by (rule) induction from the course with new cases without reproving the ones from the course, again provided you clearly state the proof you are extending.

(a) Given domains \( D_1, D_2, E_1 \) and \( E_2 \), and continuous functions \( f_1 : D_1 \to E_1 \) and \( f_2 : D_2 \to E_2 \), show that

\[
(f_1 \times f_2) : D_1 \times D_2 \to E_1 \times E_2
\]

is continuous. \([6 \text{ marks}]\)

We wish to extend PCF with the product type \( \tau_1 \times \tau_2 \), by adding the new terms \( \text{fst}, \text{snd} \) and \( \text{pair} \) to the language, such that

\[
\begin{align*}
\Gamma \vdash t : \tau_1 \times \tau_2 & & \Gamma \vdash t : \tau_1 \times \tau_2 \\
\Gamma \vdash \text{fst}(t) : \tau_1 & & \Gamma \vdash \text{snd}(t) : \tau_2 \\
\Gamma \vdash \text{pair}(t_1, t_2) : \tau_1 \times \tau_2
\end{align*}
\]

and with the following operational semantics:

\[
\begin{align*}
t \Downarrow_{\tau_1 \times \tau_2} & \quad \text{pair}(v_1, v_2) \\
\text{fst}(t) \Downarrow_{\tau_1} & \quad \text{snd}(t) \Downarrow_{\tau_2} \\
t \Downarrow_{\tau_1} & \quad t_1 \Downarrow_{\tau_1} \\
t \Downarrow_{\tau_2} & \quad t_2 \Downarrow_{\tau_2}
\end{align*}
\]

(b) Give a denotational semantics for the product type \( [\tau_1 \times \tau_2] \) in terms of \( [\tau_1] \) and \( [\tau_2] \). \([2 \text{ marks}]\)

(c) Give a denotational semantics for \( \text{fst}, \text{snd} \) and \( \text{pair} \), extending the semantics of PCF from the lectures, and justify why this semantics is well-defined according to the typing rules given above. \([6 \text{ marks}]\)

(d) Recall what it means for denotational semantics to be sound. Show that the semantics you have just given is sound. \([6 \text{ marks}]\)