12 Randomised Algorithms (tms41)

Given an undirected graph \( G = (V, E) \), an independent set \( I \subseteq V \) is a subset such that for any two vertices \( u \in I, v \in I \), there is no edge \( \{u, v\} \in E(G) \). Let \( \alpha(G) \) denote the size of the largest independent set in \( G \).

(a) Consider the following randomised algorithm for computing an independent set, which takes as input an undirected graph \( G = (V, E) \) and a fixed parameter \( p \in [0, 1] \):

Step 1: Starting with an empty set \( S \), add each vertex from \( V(G) \) to \( S \) independently with probability \( p \).

Step 2: Go through all edges \( e = \{u, v\} \in E(G) \), and for any edge \( e \) which had both vertices in \( S \) after Step 1, remove \( u \) and \( v \) from \( S \).

(i) Justify briefly why the output \( S \) of this algorithm is an independent set.

(ii) Is the output \( S \) necessarily maximal, i.e., it is not possible to add any vertex \( u \in V \) to \( S \) and obtain a larger independent set? Justify your answer.

(iii) Prove that the expected size of the output \( S \) after the second step of the algorithm is \( p \cdot |V| - p^2 \cdot |E| \).

(iv) How would you choose \( p \) in order to maximise the expected size of \( S \), as computed in (a)(iii)?

(v) What does your answer in (a)(iv) imply for \( \alpha(G) \)? Justify your answer.

(b) Formulate the problem of finding the largest independent set as an Integer Program (I), and describe the Linear Programming Relaxation (L).