12 Randomised Algorithms (tms41)

Given an undirected graph $G = (V, E)$, an **independent set** is a subset $I \subseteq V$ such that for any two vertices $u \in I, v \in I$, there is no edge $\{u, v\} \in E(G)$. Let $\alpha(G)$ denote the size of the **largest** independent set in G.

(a) Consider the following randomised algorithm for computing an independent set, which takes as input an undirected graph $G = (V, E)$ and a fixed parameter $p \in [0, 1]$:

Step 1: Starting with an empty set S, add each vertex from $V(G)$ to S independently with probability p.

Step 2: Go through all edges $e = \{u, v\} \in E(G)$, and for any edge e which had both vertices in S after Step 1, remove u **or** v from S.

(i) Justify briefly why the output S of this algorithm is an independent set. [2 marks]

(ii) Is the output S necessarily maximal, i.e., it is not possible to add any vertex $u \in V$ to S and obtain a larger independent set? Justify your answer. [3 marks]

(iii) Prove that the expected size of the output S after the second step of the algorithm is at least $p \cdot |V| - p^2 \cdot |E|$. [4 marks]

(iv) How would you choose p in order to maximise the expected size of S, as computed in (a)(iii)? [4 marks]

(v) What does your answer in (a)(iv) imply for $\alpha(G)$? Justify your answer. [3 marks]

(b) Formulate the problem of finding the largest independent set as an Integer Program (**I**), and describe the Linear Programming Relaxation (**L**). [4 marks]