4 Denotational Semantics (mgapb2)

In all parts of this question, you are allowed to use theorems from the course, provided you state them precisely beforehand.

Define the *smash product* $D \otimes E$ of two domains D and E to be the set

$$\{(x,y) \in D \times E \mid x \neq \bot_D \wedge y \neq \bot_E\} \cup \{\bot_{D\otimes E}\}$$

where $\bot_{D\otimes E}$ is a new element which is not a pair. This set is equipped with the order $\sqsubseteq_{D\otimes E}$ such that $\bot_{D\otimes E} \sqsubseteq_{D\otimes E} z$ for any z, and $(x,y) \sqsubseteq_{D\otimes E} (x',y')$ if and only if $x \sqsubseteq_D x'$ and $y \sqsubseteq_E y'$.

(*a*) Show that the smash product of two domains is a domain. [4 marks]

(*b*) Given three domains D, E and F, we call a function $f \in D \times E \to F$ *bistrict* if for any $x \in D$ and $y \in E$, $f(\bot, y) = \bot$ and $f(x, \bot) = \bot$.

Show that not all strict functions are bistrict. [3 marks]

(*c*) Let D, E and F be domains, and $f : D \times E \to F$ a function. Give a condition on the currying $\mathrm{cur}(f) : D \to (E \to F)$ of f that is necessary and sufficient for f to be bistrict. [4 marks]

(*d*) We define the function smash as follows:

$$\begin{array}{rrcll} \text{smash}: & D \times E & \to & D \otimes E & \\ & (x,y) & \mapsto & (x,y) & \text{if } x \neq \bot \text{ and } y \neq \bot \\ & (x,y) & \mapsto & \bot_{D\otimes E} & \text{otherwise} \end{array}$$

Show that if $f\colon D \times E \to F$ is continuous and bistrict, then there exists a unique $\tilde{f}\colon D \otimes E \to F$ that is strict and continuous and such that $f = \tilde{f} \circ \text{smash}$. [4 marks]

(*e*) Give the definition of $X_\bot$, the flat domain on a set X. [1 mark]

Given two sets S and T, show that the domains $(S \times T)_\bot$ and $S_\bot \otimes T_\bot$ are isomorphic, *i.e.* that there exist strict continuous functions $f\colon (S \times T)_\bot \to S_\bot \otimes T_\bot$ and $g\colon S_\bot \otimes T_\bot \to (S \times T)_\bot$ such that $f \circ g = \mathrm{id}$ and $g \circ f = \mathrm{id}$. [4 marks]