12 Randomised Algorithms (tms41)

Consider the allocation of n balls into n bins, both labelled $[n] := \{1, 2, \ldots, n\}$. We assume each ball is assigned to a bin chosen uniformly and independently at random.

(a) For any given bin, what is the expectation and variance of its load? [2 marks]

(b) What is known about the maximum load across all n bins? A proof or justification is not required here. [2 marks]

Assume now that each ball $j \in [n]$ has a random processing time B_j, which has a mean one exponential distribution, i.e., for any $t \geq 0$, $\mathbf{P}[\, B_j \geq t \,] = e^{-t}$. For a bin $i \in [n]$, let T_i be the sum of the processing times of balls allocated to i.

(c) Show that $\mathbf{E}[\, T_i \,] = 1$ for every bin $i \in [n]$. For full marks, your answer should include a justification and a formal definition of T_i. [4 marks]

(d) Find a constant $c > 0$ such that the probability that a fixed ball has processing time at least $c \cdot \log n$ is at least $n^{-1/2}$? [2 marks]

(e) Using part (d), argue that with high probability, at least one ball has a processing time of at least $c \cdot \log n$. [4 marks]

(f) Let $B := \sum_{j=1}^{n} B_j$ be the total processing time of all n balls. Prove a Chernoff Bound of the form $\mathbf{P}[\, B \geq (1+\delta) \cdot \mathbf{E}[\, B \,] \,]$, for any $\delta > 0$.
Hints: You may use the fact that for Z being exponentially distributed with mean 1, it holds for any $0 < \lambda < 1$ that $\mathbf{E}\left[\, e^{\lambda \cdot Z} \,\right] \leq \frac{1}{1-\lambda}$. Also you may want to choose $\lambda = \frac{\delta}{1+\delta}$ when optimising the tail bound. [6 marks]