Let $P$ be any register machine program.

(a) What is the partial function $f: \mathbb{N} \rightarrow \mathbb{N}$ of one argument computed by $P$? [2 marks]

(b) What is the partial function $g: \mathbb{N}^2 \rightarrow \mathbb{N}$ of two arguments computed by $P$? [2 marks]

(c) Describe the construction of a Gödel numbering of register machine programs. That is, a bijection $G$ between the natural numbers $\mathbb{N}$ and the collection of register machine programs. [5 marks]

We now write $\phi_i: \mathbb{N} \rightarrow \mathbb{N}$ for the partial function of one argument that is computed by the register machine program $G(i)$, and $\psi_i: \mathbb{N}^2 \rightarrow \mathbb{N}$ for the partial function of two arguments that is computed by the register machine program $G(i)$, where $G$ is the Gödel numbering constructed in part (c).

(d) Show that the partial function $u: \mathbb{N}^2 \rightarrow \mathbb{N}$ defined by $u(i,x) = \phi_i(x)$ for all $i$ and $x$ is computable. You may assume standard results about register machine programs, as long as you state them in full and clearly. [5 marks]

(e) Sketch a proof to show that there is a computable partial function $s: \mathbb{N}^2 \rightarrow \mathbb{N}$ such that, for all $x,y,z \in \mathbb{N}$:

$$\phi_{s(x,y)}(z) = \psi_x(y,z).$$

[6 marks]