Regular expressions are defined by the following grammar:

\[
\begin{align*}
    r &::= c & \text{Matches the single-character word } c \\
    &| \epsilon & \text{Matches the empty word} \\
    &| r_1 \circ r_2 & \text{Matches the concatenation of an } r_1\text{-word and an } r_2\text{-word} \\
    &| 0 & \text{Matches no words} \\
    &| r_1 + r_2 & \text{Matches any } r_1\text{-word or } r_2\text{-word} \\
    &| r* & \text{Matches the concatenation of a finite number of } r\text{-words}
\end{align*}
\]

(a) Give a set of inference rules defining a relation for when a word \( w \) is matched by a regular expression \( r \). Use the notation \( w \cdot w' \) to denote concatenation. [8 marks]

(b) (i) Using the matching relation defined above, define a suitable notion of semantic equivalence \( r_1 \simeq r_2 \) for regular expressions. [4 marks]

(ii) Use this definition to prove that \( (r + r') \simeq (r' + r) \). You may use inversion lemmas without proof, as long as they are explicitly indicated. [4 marks]

(c) Define an inductive relation \( r \text{ null} \) characterizing the regular expressions \( r \) for which \( \epsilon \) in \( r \). [4 marks]