9 Discrete Mathematics (js2878)

(a) Let $f\colon A \to B$ be a function.

(i) What does it mean for a function $s\colon B \to A$ to be a ***section*** of the function $f\colon A \to B$? [2 marks]

(ii) What does it mean for a function $r\colon B \to A$ to be a ***retraction*** of the function $f\colon A \to B$? [2 marks]

(b) Let $f\colon A \to B$ be a function.

(i) Let $s\colon B \to A$ be a section of $f\colon A \to B$. Prove that any two sections $u, v\colon A \to B$ of $s\colon B \to A$ are equal. [2 marks]

(ii) Let $r\colon B \to A$ be a retraction of $f\colon A \to B$. Prove that any two retractions $u, v\colon A \to B$ of $r\colon B \to A$ are equal. [2 marks]

(c) We shall refer to a given function $f\colon A \to B$ as ***locally subsingleton*** when for every $b \in B$, the inverse image $f^{-1}\{b\} \subseteq A$ has *at most* one element, *i.e.* for any $x, y \in f^{-1}\{b\}$ we have $x = y$. Prove that a function $f\colon A \to B$ is *locally subsingleton* if and only if it is *injective*. [4 marks]

(d) We shall refer to a given function $f\colon A \to B$ as ***locally singleton*** when for every $b \in B$, the inverse image $f^{-1}\{b\} \subseteq A$ has *exactly* one element.

(i) Prove that any function $f\colon A \to B$ is *locally singleton* if and only if it is *bijective*. [4 marks]

(ii) Prove that the set of *bijective functions* from A to B is itself in bijection with the set of triples (f, g, h) with $f\colon A \to B$ and $g, h\colon B \to A$ such that g is a section of f and h is a retraction of f. You may use any standard results provided that you state them clearly. [4 marks]