8 Discrete Mathematics (mpf23)

(*a*) Let $R \subseteq A \times B$ be a relation from a set A to a set B. For a subset $X \subseteq A$, define the subset $\mathrm{C}_R(X) \subseteq B$ as

$$\mathrm{C}_R(X) = \{\, b \in B \mid \forall\, a \in X.\ a\, R\, b \,\}$$

(*i*) Prove that, for all $P \subseteq Q \subseteq A$, one has $\mathrm{C}_R(Q) \subseteq \mathrm{C}_R(P)$. [2 marks]

Let $\mathcal{F} \subseteq \mathcal{P}(A)$ be a family of subsets of the set A.

(*ii*) Define the big intersection $\bigcap \mathcal{F} \subseteq A$ of $\mathcal{F}$. [2 marks]

(*iii*) Define the big union $\bigcup \mathcal{F} \subseteq A$ of $\mathcal{F}$. [2 marks]

(*iv*) Prove that $\mathrm{C}_R\big(\bigcup \mathcal{F}\big) = \bigcap \big\{\, Y \subseteq B \bigm| \exists\, X \in \mathcal{F}.\ Y = \mathrm{C}_R(X) \,\big\}$ [6 marks]

(*b*) Let A^* be the set of strings over an alphabet A.

Consider the subset W of A^* inductively defined by the following axiom and rule

$$\frac{}{\varepsilon} \qquad\qquad \frac{w}{w\,a}\ (a \in A)$$

(*i*) State the rule-induction proof method to show $\forall\, w \in W.\ P(w)$ for a property $P(w)$ of elements w of W. [2 marks]

(*ii*) Prove that $\forall\, w \in W.\ (\,\forall\, u, v \in A^*.\ w\,u = w\,v \implies u = v\,)$ by the rule-induction proof method. [6 marks]