10 Discrete Mathematics (js2878)

(*a*) Let $\Sigma = \mathbb{N}$ be the alphabet whose symbols are given by natural numbers $0, 1, 2, \ldots$, and let L_n be the language over Σ consisting of finite strings of digits that sum to n.

(*i*) Draw a diagram of a DFA recognising L_3. [3 marks]

(*ii*) For arbitrary $n \in \mathbb{N}$, define a DFA $M_n = (Q_n, \Sigma, \Delta_n, s_n, F_n)$ over $\Sigma = \mathbb{N}$ that recognises the language L_n. [4 marks]

(*b*) Let $M = (Q, \Sigma, \Delta, s, F)$ be an NFA. We shall refer to M as ***complete*** when for any $q \in Q$ and $a \in \Sigma$, there exists some $q' \in Q$ such that $q \xrightarrow{a} q'$.

We shall refer to M as ***partially deterministic*** when for any states $q, q', q'' \in Q$ and input symbol $a \in \Sigma$, if both $q \xrightarrow{a} q'$ and $q \xrightarrow{a} q''$ then $q' = q''$ holds.

(*i*) Prove that an NFA is *deterministic* if and only if it is both *complete* and *partially deterministic*. [2 marks]

(*ii*) Suppose that M is a *partially deterministic* NFA; construct a DFA $M' = (Q', \Sigma, \Delta', s', F')$ over the same alphabet such that any finite string $u \in \Sigma^*$ is accepted by M if and only if it is accepted by M' and, moreover, such that the cardinality of Q' is bounded by $\#Q' < 2^{\#Q}$ assuming $\#Q > 1$. Argue that M' meets these requirements. [8 marks]

(*c*) Let $M = (Q, \Sigma, \Delta, s, F)$ and $M' = (Q', \Sigma, \Delta', s', F')$ be two DFAs over the same alphabet, writing $\delta \colon Q \times \Sigma \to Q$ and $\delta' \colon Q' \times \Sigma \to Q'$ for the next-state functions corresponding to the total functional relations Δ and Δ' respectively.

A ***homomorphism of DFAs*** from M to M', written $f \colon M \to M'$, is defined to be a function $f \colon Q \to Q'$ satisfying the following conditions:

- f preserves the starting state, *i.e.* $fs = s'$;
- f sends accepting states to accepting states, *i.e.* for $q \in F$ we have $fq \in F'$;
- f preserves transitions, *i.e.* $f(\delta(q, a)) = \delta'(f(q), a)$ for all $q \in Q$ and $a \in \Sigma$.

(*i*) Let $M_1 = (Q_1, \Sigma, \Delta_1, s_1, F_1)$ and $M_2 = (Q_2, \Sigma, \Delta_2, s_2, F_2)$ be two DFAs over Σ. Define a new DFA $M_1 \times M_2$ over Σ with states in the cartesian product $Q_1 \times Q_2$, such that the projections $\pi_1 \colon Q_1 \times Q_2 \to Q_1$ and $\pi_2 \colon Q_1 \times Q_2 \to Q_2$ form homomorphisms $\pi_1 \colon M_1 \times M_2 \to M_1$ and $\pi_2 \colon M_1 \times M_2 \to M_2$ of DFAs, with proof. [3 marks]