9 Algorithms 2 (djw1005)

We are given a directed graph with edge costs; let $c(u \to v) > 0$ be the cost of edge $u \to v$. We are also given a start vertex s and an end vertex t, and we assume that t is reachable from s. An edge is said to be a *bottleneck* if increasing its cost results in an increase in the distance from s to t, and it is said to be an *opportunity* if decreasing its cost results in a decrease in that distance.

(*a*) For each of the following claims, either prove it or provide a counterexample:

(*i*) The graph must have an opportunity;

(*ii*) The graph must have a bottleneck;

(*iii*) All bottlenecks are opportunities.

[6 marks]

(*b*) For each edge $u \to v$, define the relaxed cost to be $c'(u \to v) = c(u \to v) + d_u - d_v$, where d_u is the distance from s to u, and d_v the distance from s to v. Prove that all opportunities have relaxed cost equal to zero. [4 marks]

(*c*) Give an algorithm for computing all opportunities in a graph g. Your algorithm should have $O(E + V \log V)$ running time, where E is the number of edges in g and V the number of vertices. Prove that your algorithm is correct, and explain why it has the desired running time. [10 marks]