8 Machine Learning and Bayesian Inference (sbh11)

You are faced with the following simple inference problem. Examples are pairs \((x, y)\), where both \(x\) and \(y\) are real numbers. You suspect that there is a relationship \(y = \theta x\) underlying the data, and that the \(y\) values are subject to additive Gaussian noise with mean 0 and variance \(\sigma^2\). You wish to infer \(\theta\) from examples \((x_i, y_i)\) where \(i = 1, \ldots, m\). You may consider \(x\) values to be fixed.

\(a\) Write down an expression for the density \(p(Y|\theta; x)\). You may use the standard expression for the Gaussian density \(N(\mu, \sigma^2)\)

\[ p(Z) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2} (Z - \mu)^2\right). \]

\([1\ \text{mark}]\]

\(b\) Write down an expression for the likelihood of the parameter \(\theta\), given the \(m\) training examples described. State any assumptions you make. \([3\ \text{marks}]\]

\(c\) Assuming that the parameter \(\theta\) has a Gaussian prior with mean 0 and variance \(\sigma_1^2\), write down an expression for the posterior density \(p(\theta|y; x)\) of \(\theta\). You may leave this expression unnormalized. State any assumptions you make. \([3\ \text{marks}]\]

\(d\) Explain how the calculation in Part \(c\) might be extended to obtain the evidence, and how this might be used to estimate the values of \(\sigma\) and \(\sigma_1\) from the training data. State any assumptions needed. You need not perform the actual calculation. \([3\ \text{marks}]\]

\(e\) Find an expression for the predictive density \(p(Y|x, x, y)\). You need not normalize the expression. You may use the identity

\[ \int_{\mathbb{R}^p} \exp\left(-\frac{1}{2} (x^T A x + x^T b + c)\right) = (2\pi)^{p/2} |A|^{-1/2} \exp\left(-\frac{1}{2} \left(c - \frac{b^T A b}{4}\right)\right). \]

\([7\ \text{marks}]\]

\(f\) What further steps are necessary should you wish to extend your result from Part \(e\) to obtain actual predictions for \(Y\) along with a measure of certainty in the prediction? \([3\ \text{marks}]\]