## 8 Machine Learning and Bayesian Inference (sbh11)

You are faced with the following simple inference problem. Examples are pairs $(x, y)$, where both $x$ and $y$ are real numbers. You suspect that there is a relationship $y = \theta x$ underlying the data, and that the $y$ values are subject to additive Gaussian noise with mean 0 and variance $\sigma^2$. You wish to infer $\theta$ from examples $(x_i, y_i)$ where $i = 1, \ldots, m$. You may consider $x$ values to be fixed.

(*a*) Write down an expression for the density $p(Y|\theta; x)$. You may use the standard expression for the Gaussian density $N(\mu, \sigma^2)$

$$p(Z) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(Z-\mu)^2\right).$$

[1 mark]

(*b*) Write down an expression for the likelihood of the parameter $\theta$, given the $m$ training examples described. State any assumptions you make. [3 marks]

(*c*) Assuming that the parameter $\theta$ has a Gaussian prior with mean 0 and variance $\sigma_1^2$, write down an expression for the posterior density $p(\theta|\mathbf{y}; \mathbf{x})$ of $\theta$. You may leave this expression unnormalized. State any assumptions you make.

[3 marks]

(*d*) Explain how the calculation in Part (*c*) might be extended to obtain the *evidence*, and how this might be used to estimate the values of $\sigma$ and $\sigma_1$ from the training data. State any assumptions needed. You need not perform the actual calculation. [3 marks]

(*e*) Find an expression for the *predictive density* $p(Y|x, \mathbf{x}, \mathbf{y})$. You need not normalize the expression. You may use the identity

$$\int_{\mathbb{R}^p} \exp\left(-\frac{1}{2}\left(\mathbf{x}^T\mathbf{A}\mathbf{x} + \mathbf{x}^T\mathbf{b} + c\right)\right) = (2\pi)^{p/2}|\mathbf{A}|^{-1/2} \exp\left(-\frac{1}{2}\left(c - \frac{\mathbf{b}^t\mathbf{A}\mathbf{b}}{4}\right)\right).$$

[7 marks]

(*f*) What further steps are necessary should you wish to extend your result from Part (*e*) to obtain actual predictions for $Y$ along with a measure of certainty in the prediction? [3 marks]