COMPUTER SCIENCE TRIPOS Part II – 2023 – Paper 9

8 Machine Learning and Bayesian Inference (sbh11)

You are faced with the following simple inference problem. Examples are pairs (x, y), where both x and y are real numbers. You suspect that there is a relationship $y = \theta x$ underlying the data, and that the y values are subject to additive Gaussian noise with mean 0 and variance σ^2 . You wish to infer θ from examples (x_i, y_i) where $i = 1, \ldots, m$. You may consider x values to be fixed.

(a) Write down an expression for the density $p(Y|\theta; x)$. You may use the standard expression for the Gaussian density $N(\mu, \sigma^2)$

$$p(Z) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(Z-\mu)^2\right).$$

- (b) Write down an expression for the likelihood of the parameter θ , given the *m* training examples described. State any assumptions you make. [3 marks]
- (c) Assuming that the parameter θ has a Gaussian prior with mean 0 and variance σ_1^2 , write down an expression for the posterior density $p(\theta|\mathbf{y}; \mathbf{x})$ of θ . You may leave this expression unnormalized. State any assumptions you make.

[3 marks]

[1 mark]

- (d) Explain how the calculation in Part (c) might be extended to obtain the evidence, and how this might be used to estimate the values of σ and σ_1 from the training data. State any assumptions needed. You need not perform the actual calculation. [3 marks]
- (e) Find an expression for the *predictive density* $p(Y|x, \mathbf{x}, \mathbf{y})$. You need not normalize the expression. You may use the identity

$$\int_{\mathbb{R}^p} \exp\left(-\frac{1}{2}\left(\mathbf{x}^T \mathbf{A} \mathbf{x} + \mathbf{x}^T \mathbf{b} + c\right)\right) = (2\pi)^{p/2} |\mathbf{A}|^{-1/2} \exp\left(-\frac{1}{2}\left(c - \frac{\mathbf{b}^t \mathbf{A} \mathbf{b}}{4}\right)\right).$$
[7 marks]

(f) What further steps are necessary should you wish to extend your result from Part (e) to obtain actual predictions for Y along with a measure of certainty in the prediction? [3 marks]