5 Denotational Semantics (mpf23)

(*a*) Consider the following definitions:

- For $L \in \mathrm{PCF}_{nat}$ and $k \in \mathbb{N}$, $L \Vdash_0 k$ if, and only if, $L \Downarrow_{nat} \mathbf{succ}^k(\mathbf{0})$.
- For $M \in \mathrm{PCF}_{nat \to nat}$ and $f : \mathbb{N} \to \mathbb{N}$, $M \Vdash_1 f$ if, and only if, for all $i \in \mathbb{N}$, $M\ \mathbf{succ}^i(\mathbf{0}) \Downarrow_{nat} \mathbf{succ}^{f(i)}(\mathbf{0})$.
- For $N \in \mathrm{PCF}_{nat \to nat \to nat}$ and $g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, $N \Vdash_2 g$ if, and only if, for all $i, j \in \mathbb{N}$, $N\ \ \mathbf{succ}^i(\mathbf{0})\ \ \mathbf{succ}^j(\mathbf{0}) \Downarrow_{nat} \mathbf{succ}^{g(i,j)}(\mathbf{0})$.

(*i*) Prove that $N \Vdash_2 g$ and $L \Vdash_0 k$ imply $N\, L \Vdash_1 \lambda\, x \in \mathbb{N}.\, g(k, x)$. [6 marks]

(*ii*) Prove that there are $N \in \mathrm{PCF}_{nat \to nat \to nat}$ and a bijection $g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that $N \Vdash_2 g$. You may use standard results provided that you state them clearly. [6 marks]

(*b*) (*i*) Say whether or not the following statement holds:

For all PCF types τ and all closed PCF terms M of type $\tau \to \tau \to \tau$, the closed PCF terms $\mathbf{fix}\big(\mathbf{fn}\ x : \tau.\, \mathbf{fix}(\mathbf{fn}\ y : \tau.\, M\, y\, x)\big)$ and $\mathbf{fix}(\mathbf{fn}\ z : \tau.\, M\, z\, z)$ of type τ are contextually equivalent. [2 marks]

(*ii*) Either prove or disprove the above statement. [6 marks]