COMPUTER SCIENCE TRIPOS Part II – 2023 – Paper 9

12 Randomised Algorithms (tms41)

Consider the following problem (**P**). We are given a *directed* graph G = (V, E) with non-negative edge-weights $w_{i,j} \ge 0$ for each edge $(i, j) \in E$. The task is to partition V into two sets V_1 and $V_2 = V \setminus V_1$ so as to maximize the total weight of edges going from V_1 and V_2 , i.e.,

maximise
$$\sum_{(i,j)\in E: i\in V_1, j\in V_2} w_{i,j}.$$

- (a) Design a randomised approximation algorithm for (\mathbf{P}) with running time O(V)and analyse its approximation ratio. [5 marks]
- (b) In this part of the question, we additionally want that $|V_1| = |V_2| = n/2$ (we assume for simplicity that n = |V| is an even integer). Adjust the algorithm and analysis from (a). [4 marks]
- (c) Consider the following integer program called (\mathbf{I}) :

maximise	$\sum_{(i,j)\in E} w_{i,j} z_{i,j}$	
subject to	$z_{i,j} \le x_i$	for each $(i, j) \in E$
	$z_{i,j} \le 1 - x_j$	for each $(i, j) \in E$
	$x_i \in \{0, 1\}$	for $i \in V$
	$z_{i,j} \in [0,1]$	for each $(i, j) \in E$

- (i) Prove that this integer program solves the problem (\mathbf{P}) . [4 marks]
- (*ii*) Consider the following randomised algorithm for (**P**). Let $(\overline{z}, \overline{x})$ be a solution to a linear program, which is identical to (**I**) but with $x_i \in \{0, 1\}$ replaced by $x_i \in [0, 1]$. We then put each vertex *i* into V_1 with probability $1/4 + \overline{x}_i/2$, independently.
 - (A) Explain briefly why this algorithm can be implemented in polynomial time. [1 mark]
 - (B) Prove that the approximation ratio of this algorithm is 2. [6 marks]