12 Randomised Algorithms (tms41)

Consider the following problem (**P**). We are given a *directed* graph $G = (V, E)$ with non-negative edge-weights $w_{i,j} \geq 0$ for each edge $(i, j) \in E$. The task is to partition V into two sets V_1 and $V_2 = V \setminus V_1$ so as to maximize the total weight of edges going from V_1 and V_2, i.e.,

$$\text{maximise} \sum_{(i,j)\in E:\, i\in V_1, j\in V_2} w_{i,j}.$$

(*a*) Design a randomised approximation algorithm for (**P**) with running time $O(V)$ and analyse its approximation ratio. [5 marks]

(*b*) In this part of the question, we additionally want that $|V_1| = |V_2| = n/2$ (we assume for simplicity that $n = |V|$ is an even integer). Adjust the algorithm and analysis from (*a*). [4 marks]

(*c*) Consider the following integer program called (**I**):

$$\begin{aligned} \text{maximise} \quad & \sum_{(i,j)\in E} w_{i,j} z_{i,j} \\ \text{subject to} \quad & z_{i,j} \leq x_i && \text{for each } (i,j) \in E \\ & z_{i,j} \leq 1 - x_j && \text{for each } (i,j) \in E \\ & x_i \in \{0,1\} && \text{for } i \in V \\ & z_{i,j} \in [0,1] && \text{for each } (i,j) \in E \end{aligned}$$

(*i*) Prove that this integer program solves the problem (**P**). [4 marks]

(*ii*) Consider the following randomised algorithm for (**P**). Let $(\overline{z}, \overline{x})$ be a solution to a linear program, which is identical to (**I**) but with $x_i \in \{0, 1\}$ replaced by $x_i \in [0, 1]$. We then put each vertex i into V_1 with probability $1/4 + \overline{x}_i/2$, independently.

(A) Explain briefly why this algorithm can be implemented in polynomial time. [1 mark]

(B) Prove that the approximation ratio of this algorithm is 2. [6 marks]