COMPUTER SCIENCE TRIPOS Part II – 2023 – Paper 8

12 Randomised Algorithms (tms41)

In this question, all considered graphs are undirected and *d*-regular.

- (a) State the definition of conductance. [2 marks]
- (b) If G is disconnected, what does this imply in terms of the conductance? [1 mark]
- (c) If G is disconnected, what does this imply in terms of the spectrum of \mathbf{L} ? Briefly justify your claim. [3 marks]
- (d) The *d*-dimensional hypercube with $n = 2^d$ vertices is defined by creating a vertex for each *d*-digit binary number $(x_1, x_2, \ldots, x_d) \in \{0, 1\}^d$. Further, any two vertices are adjacent if and only if their binary representations differ in exactly one digit.
 - (i) Identifying each binary representation $(x_1, x_2, \ldots, x_d) \in \{0, 1\}^d$ with a unique vertex label in $\{1, 2, \ldots, n\}$, verify that $f_{(x_1, x_2, \ldots, x_d)} = (-1)^{x_1}$ is an eigenvector of the Laplacian Matrix **L** of the hypercube. State the associated eigenvalue of f for both **L** and the adjacency matrix **A**. [7 marks]
 - (*ii*) Apply the Spectral Clustering Algorithm to estimate the conductance of the hypercube, assuming that f in (d)(i) is the eigenvector of λ_2 . [*Hint:* It suffices to apply the (n/2)-th sweep cut only.] [5 marks]
 - (*iii*) Combining (d)(ii) and (d)(i), what can you conclude about the found cut? [2 marks]