9 Discrete Mathematics (mpf23)

(a) Let $B \subseteq \{<, >\}^\ast$ be the set inductively defined by the axiom and rule below

\[
\begin{array}{c|c}
< > & \frac{\ell}{< \ell \ r >} \\
\end{array}
\]

and let $f : B \to B$ be the inductively defined function given by \( f(<>) = < > \), \( f(<\ell \ r>) = < f(r) \ f(\ell) > \)

(i) State whether or not $f$ is the identity function on $B$, and prove your claim. [2 marks]

(ii) State whether or not $f$ is a bijection, and prove your claim. [5 marks]

(b) Let $L \subseteq \{a\}^\ast \times \mathbb{N}$ be the relation inductively defined by the axiom and rule below

\[
\begin{array}{c|c}
(a, 1) & \frac{(u,m) \ (v,n)}{(uv,m+n)} \\
\end{array}
\]

(i) Give a pair in $\{a\}^\ast \times \mathbb{N}$ together with two different derivations that show that the pair is in $L$. [2 marks]

(ii) Prove that, for all $(w,k) \in L$, $k \geq 1$. [5 marks]

(iii) Prove that, for all $n \in \mathbb{N}$, $(\epsilon,n) \notin L$. [6 marks]

[Hint: Argue by contradiction.]