8 Discrete Mathematics (mpf23)

(*a*) (*i*) Show that $(x-1)$ divides $(x^n - 1)$ for all positive integers x and n. [3 marks]

(*ii*) A positive integer n is said to be composite whenever there are positive integers a and b greater than 1 such that $n = a \cdot b$.

Prove that, for all positive integers x greater than 1, if a positive integer n is composite then so is $x^n - 1$. [3 marks]

[*Hint*: Consider the instance of the above statement for $x = 2$.]

(*b*) Prove that, for all natural numbers n, $24 \mid (2 \cdot 7^n - 3 \cdot 5^n + 1)$. [6 marks]

[*Hint*: Note that $7^2 \equiv 1 \pmod{24}$ and $5^2 \equiv 1 \pmod{24}$. Consider using the principle of strong mathematical induction.]

(*c*) Say whether each of the following statements is true or false, and prove your claim.

(*i*) For all sets A and B, and all functions f and g from A to $\mathcal{P}(B)$,

$$\big[\,\forall\, a \in A.\ \exists\, x \in A.\ f(a) \subseteq g(x)\,\big] \ \Rightarrow\ \bigcup_{a \in A} f(a) \subseteq \bigcup_{x \in A} g(x)$$

[4 marks]

(*ii*) For all sets A and B, and all functions f and g from A to $\mathcal{P}(B)$,

$$\bigcup_{a \in A} f(a) \subseteq \bigcup_{x \in A} g(x) \ \Rightarrow\ \big[\,\forall\, a \in A.\ \exists\, x \in A.\ f(a) \subseteq g(x)\,\big]$$

[4 marks]