6 Introduction to Probability (mj201+tms41)

(a) Let \( X \sim \text{Uni}(0, 1/2) \) be a uniform continuous random variable. What are \( E[X] \) and \( \text{Var}[X] \)?

(b) Let \( X \sim \text{Uni}(0, 1/2) \) and \( Y \sim \text{Uni}(0, 1/2) \) be two independent uniform continuous random variables, and define \( Z = \min(X, Y) \).

(i) What is the cumulative distribution function of \( Z \)?

(ii) What is \( E[Z] \)?

(c) Let \( X \sim \text{Uni}(0, 1) \) and \( Y \sim \text{Uni}(0, 1) \) be two independent uniform continuous random variables.

(i) Consider a random triangle between the three points \((0, 0), (1, 0)\) and \((X, Y)\), as illustrated in the figure above. What is the expectation of the area?

(ii) Now consider a random circle with center \((X, Y)\) such that the circumference is as large as possible but remains within the unit-square \([0, 1]^2\) (see figure). What is the expectation of the circumference?

(iii) Based on your answer from (c)(ii), what can you conclude about the expectation of the area of this circle?

(iv) Additionally, let \( X' \sim \text{Uni}(0, 1) \) and \( Y' \sim \text{Uni}(0, 1) \) be two uniform continuous random variables and assume \( X, Y, X', Y' \) are mutually independent. Consider a random rectangle with corner points \((X, Y)\) and \((X', Y')\), which are diagonally opposite. What is the expectation of the circumference?