5 Introduction to Probability (mj201+tms41)

We consider the round of the last eight in the UEFA football playoffs, which involves a pairing of eight teams into four matches. Assume that these pairings are decided randomly by a lottery. Further, assume that four of the eight teams are considered *strong* and the other four are considered *weak*.

(*a*) What is the expected number of matches between a strong team and a weak team? [2 marks]

(*b*) What is the probability that all four matches are between a strong and a weak team? [3 marks]

Each match consists of a home game and an away game, and for each match the team which plays home at the first game is decided uniformly and independently at random.

(*c*) Let a random variable Z be the number of strong teams which play home at the first game.

(*i*) Determine the distribution of Z. [2 marks]

(*ii*) Compute the probability that $\mathbf{P}[Z \geq 1]$. [2 marks]

It turns out that matches between two strong teams are more likely to go into extra time (which can only occur in the second of the two games). Specifically, if both teams are strong, then this happens with probability 1/3; if exactly one team is weak, then this happens with probability 1/4; and finally, if both teams are weak, then this happens with probability 1/4, too.

(*d*) Consider one match, and let random variable $S \in \{0, 1, 2\}$ be the number of strong teams in that match, and let random variable $Y \in \{0, 1\}$ be an indicator which is 1 if and only if the second game of the match goes into extra time.

(*i*) Compute $\mathbf{E}[S]$ and $\mathbf{E}[Y]$. [2 marks]

(*ii*) Compute $\mathbf{Cov}[S, Y]$. [3 marks]

(*iii*) Are S and Y independent? Justify your answer. [2 marks]

(*e*) Not knowing the pairings, somebody tells you how many of the four matches went to extra time. How would you estimate the number of matches between two strong teams? [4 marks]