4 Denotational Semantics (mpf23)

(*a*) For posets D and E, and for monotone functions $f : D \to E$ and $g : E \to D$, we write $f : D \cong E : g$ whenever $g \circ f = \mathrm{id}_D$ and $f \circ g = \mathrm{id}_E$. Moreover, we say that D and E are *isomorphic*, and write $D \cong E$, whenever $f : D \cong E : g$ for some f and g.

Prove that for domains D and E, and for monotone functions $f : D \to E$ and $g : E \to D$, if $f : D \cong E : g$ then f and g are continuous. [4 marks]

(*b*) A *refsym* is defined to be a pair $\underline{A} = (A, \sim_A)$ consisting of a set A together with a binary relation on it $\sim_A \subseteq A \times A$ that is reflexive (namely, $x \sim_A x$ for all $x \in A$) and symmetric (namely, $x \sim_A y$ implies $y \sim_A x$ for all $x, y \in A$).

For a refsym $\underline{A} = (A, \sim_A)$, define $\Delta(\underline{A}) = \{ \alpha \subseteq A \mid \forall x, y \in \alpha.\ x \sim_A y \}$.

(*i*) Prove that for a refsym $\underline{A}$, the pair $\mathrm{D}(\underline{A}) = (\Delta(\underline{A}), \subseteq)$ is a domain. [5 marks]

(*ii*) For a set A, define a refsym $F(A)$ such that the domain $\mathrm{D}(F(A))$ and the flat domain $A_\perp$ are isomorphic. Establish the isomorphism $\mathrm{D}(F(A)) \cong A_\perp$. [5 marks]

(*iii*) For refsyms $\underline{A}_1$ and $\underline{A}_2$ define a refsym $P(\underline{A}_1, \underline{A}_2)$ such that the domain $\mathrm{D}(P(\underline{A}_1, \underline{A}_2))$ and the product domain $\mathrm{D}(\underline{A}_1) \times \mathrm{D}(\underline{A}_2)$ are isomorphic. Establish the isomorphism $\mathrm{D}(P(\underline{A}_1, \underline{A}_2)) \cong \mathrm{D}(\underline{A}_1) \times \mathrm{D}(\underline{A}_2)$. [6 marks]