12 Randomised Algorithms (tms41)

(*a*) Consider the following Markov chain with state space $\Omega = \{1, 2\}$ and transition matrix:

$$P = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix},$$

where $p \in [0, 1]$ and $q \in [0, 1]$.

(*i*) For the class of Markov chain above, state whether an instance: (1) is irreducible, (2) is aperiodic and (3) has a unique stationary distribution. Pay attention to special cases. [8 marks]

(*b*) Consider now the transition matrix:

$$P = \begin{pmatrix} 5/6 & 1/6 \\ 1/3 & 2/3 \end{pmatrix}.$$

(*i*) Prove that for any integer $k \geq 1$, the k-th power of P satisfies:

$$P^k = \begin{pmatrix} 2/3 & 1/3 \\ 2/3 & 1/3 \end{pmatrix} + (1/2)^{k-1} \cdot \begin{pmatrix} 1/6 & -1/6 \\ -1/3 & 1/3 \end{pmatrix}$$

[4 marks]

(*ii*) State the general definition of the mixing time $\tau(\epsilon)$ of a Markov chain with transition matrix P. [2 marks]

(*iii*) Consider now again the transition matrix P from (b). What can you deduce for $\tau(1/24)$?
Hint: You may use the formula from $(b)(i)$. [6 marks]