6 Computation Theory (amp12)

(a) Explain why the Church-Rosser Theorem implies that any $\lambda$-term that is $\beta$-convertible ($\equiv_\beta$) to a term in $\beta$-normal form is in fact $\beta$-reducible ($\rightarrow$) to one in $\beta$-normal form. [2 marks]

(b) Let $Bnf$ denote the set of $\lambda$-terms that have a $\beta$-normal form. Give with justification an example of two closed $\lambda$-terms $I$ and $\Omega$ with $I \in Bnf$ and $\Omega \notin Bnf$. [3 marks]

(c) Suppose that $#$ is a bijection between the set of all $\lambda$-terms and the set $\mathbb{N}$ of all natural numbers and that there are recursive functions $\alpha : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ and $\nu : \mathbb{N} \rightarrow \mathbb{N}$ satisfying for all $\lambda$-terms $M, N$ and numbers $n$ that

\[
\alpha (#(M), #(N)) = # (MN) \quad (1)
\]
\[
\nu (n) = # (n) \quad (2)
\]

where the $\lambda$-term $n$ is the $n^{th}$ Church numeral. Writing $\Gamma M^n$ for $\#(M)$, show that there are closed $\lambda$-terms $\text{App}$ and $\text{Num}$ satisfying for all $\lambda$-terms $M, N$ that

\[
\text{App} \Gamma M^n \Gamma N^n = \beta \Gamma M N^n \quad (3)
\]
\[
\text{Num} \Gamma N^n = \beta \Gamma N N^n \quad (4)
\]

(Any general properties of partial recursive functions with respect to $\lambda$-calculus you use should be carefully stated, but need not be proved.) [6 marks]

(d) Consider the following property of a closed $\lambda$-term $F$ (where $\Gamma M^n$ is as in part (c));

for all $\lambda$-terms $M$,

\[
\begin{cases}
F \Gamma M^n = \beta 0 & \text{if } M \in Bnf \\
F \Gamma M^n = \beta 1 & \text{if } M \notin Bnf
\end{cases} \quad (7)
\]

Let $H = \lambda h. P. \Omega 1 (F (\text{App} h (\text{Num} h)))$ where $P = \lambda x y f. f (\lambda z. y) x$ and $\Omega$ and $1$ are as in part (b). By considering whether or not $H \Gamma H^n$ is in $Bnf$, deduce that there can be no $\lambda$-term $F$ satisfying (7). [6 marks]

(e) Deduce from part (d) that $\{ \#(M) \mid M \in Bnf \}$ is an undecidable set of numbers. [3 marks]