9 Discrete Mathematics (mpf23)

(a) For sets $A$ and $B$, recall that $A \Rightarrow B$ denotes the set of all functions from $A$ to $B$ and that $f : A \rightarrow B$ states that $f$ is a function from $A$ to $B$.

(i) Let $R$ be a set.

For a set $X$ define $\eta_X : X \rightarrow ((X \Rightarrow R) \Rightarrow R)$ by

$$\eta_X(x)(f) = f(x)$$

and define $F : (((X \Rightarrow R) \Rightarrow R) \Rightarrow R) \rightarrow (X \Rightarrow R)$ by

$$F(\varphi)(x) = \varphi(\eta_X(x))$$

Prove that $F$ is surjective. [Hint: $F$ is actually a retraction.] [6 marks]

(ii) Using the above, or otherwise, prove that for all sets $X$ and $R$, if there is a surjection from $X$ to $((X \Rightarrow R) \Rightarrow R)$ then $R$ is a singleton. You may use standard results provided that you state them clearly. [4 marks]

(b) For sets $\Sigma$ and $A$, let $a \in A$ and $f : \Sigma \times \Sigma^* \times A \rightarrow A$. Let $R$ be the subset of $\Sigma^* \times A$ inductively defined by the axiom

$$((\varepsilon, a))$$

and the rule

$$\frac{(w, x)}{(sw, f(s, w, x))} \quad (s \in \Sigma, w \in \Sigma^*, x \in A)$$

Prove that:

(i) $R$ is total; that is, $\forall w \in \Sigma^*. \exists x \in A. (w, x) \in R.$ [4 marks]

(ii) $R$ is functional; that is, $\forall (w, x) \in R. \forall y \in A. (w, y) \in R \Rightarrow y = x.$ [6 marks]