8 Discrete Mathematics (mpf23)

(a) Without using the Fundamental Theorem of Arithmetic, prove that for all positive integers \(a, b, c\), if \(\text{gcd}(b, c) = 1\) then \(\text{gcd}(a, b \cdot c) = \text{gcd}(a, b) \cdot \text{gcd}(a, c)\). [4 marks]

(b) Let \(k\) be a fixed integer.

Set \(p_0 = q_0 = 1\) and, for \(n \in \mathbb{N}\), let \(p_{n+1} = p_n + k q_n\) and \(q_{n+1} = p_n + q_n\).

For \(n \in \mathbb{N}\), define \(r_n = |k(q_n)^2 - (p_n)^2|\)

(i) For \(n \in \mathbb{N}\), give a closed-form expression \(s_n\) defined in terms of \(k\) and \(n\) such that \(s_n = r_n\). [3 marks]

(ii) Prove that \(s_n = r_n\) for all \(n \in \mathbb{N}\). [5 marks]

(c) Fix sets \(A\) and \(B\).

Consider a set \(P\) together with functions \(p : P \to A\) and \(q : P \to B\) such that for all sets \(X\) and for all functions \(f : X \to A\) and \(g : X \to B\) there exists a unique function \(u(f, g) : X \to P\) satisfying \(p \circ u(f, g) = f\) and \(q \circ u(f, g) = g\).

(i) Define a function from \(P\) to the product \(A \times B\). [1 mark]

(ii) Define a function from the product \(A \times B\) to \(P\). [1 mark]

(iii) Prove that \(u(p, q) : P \to P\) is the identity function. [2 marks]

(iv) Prove that \(P\) and the product \(A \times B\) are isomorphic. [4 marks]