

5 Introduction to Probability (mj201)

A student is taking a multiple-choice exam consisting of n questions. Assume for each question, the student either knows the correct answer or guesses one of the four available answers randomly. Further, assume that for each question, the student knows the correct answer with probability $p = 0.6$, and these events are independent across different questions.

- (a) Let Z be the random variable counting the correct answers for the n questions given by the student. What is the distribution of Z ? Also state its expectation and variance. [4 marks]
- (b) Assuming $n \rightarrow \infty$, how could you approximate the distribution of Z in order to estimate $\mathbf{P}[Z \geq x]$ for some $x \geq 0$? [4 marks]
- (c) Given that the student answers a question correctly, what is the probability that the student actually knows the answer? [4 marks]
- (d) Suppose we know that the student will answer all n questions of the exam correctly with probability 0.343. What is the number of questions n on the exam? [2 marks]

Consider now a refined probabilistic model with $n = 2$ questions. For $i \in \{1, 2\}$, let X_i be an indicator random variable which is 1 iff question i is answered correctly by the student. We know $\mathbf{P}[X_1 = 1 \mid X_2 = 0] = 0.4$, $\mathbf{P}[X_2 = 1 \mid X_1 = 1] = 0.8$ and $\mathbf{P}[X_1 = 1] = 0.5$.

- (e) Compute the joint distribution of (X_1, X_2) . [4 marks]
- (f) Are X_1 and X_2 independent? [2 marks]